

Semiintegrable almost Grassmann structures*

M.A. Akivis¹

*Department of Mathematics, Jerusalem College of Technology – Mahon Lev, 21 Havaad, Haleumi St.,
P.O.B. 16031, Jerusalem 91160, Israel*

V.V. Goldberg²

Department of Mathematics, New Jersey Inst. of Technology, University Heights, Newark, NJ 07102, USA

Communicated by A. Gray

Received 1 September 1997

Abstract: In the present paper we study locally semiflat (we also call them semiintegrable) almost Grassmann structures. We establish necessary and sufficient conditions for an almost Grassmann structure to be α - or β -semiintegrable. These conditions are expressed in terms of the fundamental tensors of almost Grassmann structures. Since we are not able to prove the existence of locally semiflat almost Grassmann structures in the general case, we give many examples of α - and β -semiintegrable structures, mostly four-dimensional. For all examples we find systems of differential equations of the families of integral submanifolds V_α and V_β of the distributions Δ_α and Δ_β of plane elements associated with an almost Grassmann structure. For some examples we were able to integrate these systems and find closed form equations of submanifolds V_α and V_β .

Keywords: Almost Grassmann structure, locally semiintegrable, locally semiflat, webs.

MS classification: 53A40, 53A60.

0. Introduction

In the paper [2] (see also the book [1, Ch. 7]) the authors constructed the real theory of almost Grassmann structures $AG(p-1, p+q-1)$ defined on a differentiable manifold of dimension $n = pq$ by a fibration of Segre cones $SC(p, q)$. In particular, in [2] we derived the structure equations of $AG(p-1, p+q-1)$ and found (in a fourth-order differential neighborhood) a complete geometric object of the almost Grassmann structure totally defining its geometric structure. We also found the structure group of these structures and its differential prolongation and the conditions under which an almost Grassmann structure is locally flat or locally semiflat.

While constructing this theory, we distinguished three cases: $p = 2, q = 2$ ($\dim M = 4$); $p = 2, q > 2$ (or $p > 2, q = 2$); and $p > 2, q > 2$. We constructed the fundamental geometric objects of these structures up to fourth order for each of these three cases and established connections among them. In the first case the almost Grassmann structure $AG(1, 3)$ is equivalent to the pseudoconformal structure $CO(2, 2)$. Since the four-dimensional conformal structures play an important role in general relativity, this provides a physical justification for studying the structures $AG(1, 3)$ as well as for studying the general almost Grassmann structures $AG(p-1, p+q-1)$.

¹ E-mail: akivis@math.jct.ac.il.

² Corresponding author. E-mail: vlgold@m.njit.edu.

* The research of the first author was partially supported by the Israel Ministry of Absorption and the Israel Public Council for Soviet Jewry. The research of the second author was partially supported by the Research Council of the Catholic University of Leuven, Leuven, Belgium.

In the present paper we study locally semiflat almost Grassmann structures (we also call them semiintegrable), and for different values p and q we establish necessary and sufficient conditions of α - and β -semiintegrability of almost Grassmann structures. These conditions are expressed in terms of the fundamental tensors of almost Grassmann structures. We find the relations between 10 independent components of an almost Grassmann structure $AG(1, 3)$ and 10 independent components of an equivalent pseudoconformal structure $CO(2, 2)$.

Since we are not able to prove the existence of locally semiflat almost Grassmann structures in the general case, we give many examples of α - and β -semiintegrable structures, mostly for $p = q = 2$. For all examples we find systems of differential equations of the families of integral submanifolds V_α and V_β of the distributions Δ_α and Δ_β of plane elements associated with almost Grassmann structures. For some examples we were able to integrate these systems and find closed form equations of submanifolds V_α and V_β .

Note that the existence of globally semiflat four-dimensional conformal structures was proved in [11] (see also [10]).

1. Almost Grassmann structures

1. First we recall the definition of Segre varieties and Segre cones. The *Segre variety* $S(k, l)$ is an embedding of the direct product $P^k \times P^l$ of projective spaces P^k and P^l of dimensions k and l into a projective space of dimension $(k + 1)(l + 1) - 1 = kl + k + l$. Analytically this embedding can be written by means of the following equations:

$$z_\alpha^i = t_\alpha s^i, \quad \alpha = 0, 1, \dots, k; \quad i = 0, 1, \dots, l, \quad (1.1)$$

where t_α , s^i , and z_α^i are homogeneous coordinates in the spaces P^k , P^l and P^{kl+k+l} , respectively. These equations are equivalent to the condition

$$\text{rank}(z_\alpha^i) = 1. \quad (1.2)$$

The Segre variety $S(k, l)$ has the dimension $k + l$. It is proved in algebraic geometry that the degree of this variety is

$$\deg S(k, l) = \binom{k+l}{k}.$$

The cone $SC_x(k + 1, l + 1)$ with vertex at the point x whose projectivization is the Segre variety $S(k, l)$ is called the *Segre cone*.

Now we can define the notion of almost Grassmann structure.

Definition 1.1. Let M be a differentiable manifold of dimension pq , let $SC(p, q)$ be a differentiable fibration of Segre cones with the base M such that $SC_x(M) \subset T_x(M)$, $x \in M$. The pair $(M, SC(p, q))$ is said to be an *almost Grassmann structure* and is denoted by $AG(p-1, p+q-1)$. The manifold M endowed with such a structure is said to be an *almost Grassmann manifold*.

Note that the almost Grassmann structure $AG(p-1, p+q-1)$ is equivalent to the structure $AG(q-1, p+q-1)$ since both of these structures are generated on the manifold M by a differentiable family of Segre cones $SC_x(p, q)$.

In [2] we discussed the following examples of almost Grassmann structures: the almost Grassmann structure associated with the Grassmannian $G(m, n)$ (in this case $p = m + 1$ and $q = n - m$); the almost Grassmann structure $AG(1, 3)$ which is equivalent to the pseudoconformal $CO(2, 2)$ -structure; and almost Grassmann associated with multidimensional webs.

2. The structural group of the almost Grassmann structure is a subgroup of the general linear group $\mathbf{GL}(pq)$ of transformations of the space $T_x(M)$, which leave the cone $SC_x(p, q) \subset T_x(M)$ invariant. We denote this group by $G = \mathbf{GL}(p, q)$.

To clarify the structure of this group, we consider in the tangent space $T_x(M)$ a family of frames $\{e_i^\alpha\}$, $\alpha = 1, \dots, p$; $i = p+1, \dots, p+q$, such that for any fixed i , the vectors e_i^α belong to a p -dimensional generator ξ of the Segre cone $SC_x(p, q)$, and for any fixed α , the vectors e_i^α belong to a q -dimensional generator η of $SC_x(p, q)$. In such a frame, the equations of the cone $SC_x(p, q)$ can be written in the form (1.1) where now $\alpha = 1, \dots, p$; $i = p+1, \dots, p+q$; z_α^i are the coordinates of a vector $z = z_\alpha^i e_i^\alpha \subset T_x(M)$, and t_α and s^i are parameters on which a vector $z \subset SC_x(M)$ depends.

As was shown in [1], the group G of admissible transformations of the frames $\{e_i^\alpha\}$ keeping the Segre cone $SC_x(p, q)$ invariant can be presented in the form:

$$G = \mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}, \quad (1.3)$$

where $\mathbf{SL}(p)$ and $\mathbf{SL}(q)$ are special linear groups in spaces of dimensions p and q , and $\mathbf{H} = \mathbb{R}^* \otimes \text{Id}$ is the group of homotheties in $T_x(M)$. It follows that *an almost Grassmann structure $AG(m, n)$ is a G -structure of first order.*

From equation (1.1) defining the Segre cone $SC_x(p, q)$ it follows that this cone carries a $(q-1)$ -parameter family of p -dimensional generators ξ and a $(p-1)$ -parameter family of q -dimensional generators η .

The p -dimensional generators ξ form a fiber bundle on the manifold M . The base of this bundle is the manifold M , and its fiber attached to a point $x \in M$ is the set of all p -dimensional plane generators ξ of the Segre cone $SC_x(p, q)$. The dimension of a fiber is $q-1$, and it is parametrized by means of a projective space P_α , $\dim P_\alpha = q-1$. We will denote this fiber bundle of p -subspaces by $E_\alpha = (M, P_\alpha)$.

In a similar manner, q -dimensional plane generators η of the Segre cone $SC_x(p, q)$ form on M the fiber bundle $E_\beta = (M, P_\beta)$ with the base M and fibers of dimension $p-1 = \dim P_\beta$. The fibers are q -dimensional plane generators η of the Segre cone $SC_x(p, q)$.

Consider the manifold $M_\alpha = M \times P_\alpha$ of dimension $pq + q - 1$. The fiber bundle E_α induces on M_α the distribution Δ_α of plane elements ξ_α of dimension q . In a similar manner, on the manifold $M_\beta = M \times P_\beta$ the fiber bundle E_β induces the distribution Δ_β of plane elements ξ_β of dimension p .

Definition 1.2. An almost Grassmann structure $AG(p-1, p+q-1)$ is said to be α -semiintegrable if the distribution Δ_α is integrable on this structure. Similarly, an almost Grassmann structure $AG(p-1, p+q-1)$ is said to be β -semiintegrable if the distribution Δ_β is integrable on this structure. A structure $AG(p-1, p+q-1)$ is called *integrable* if it is both α - and β -semiintegrable.

Integral manifolds \tilde{V}_α of the distribution Δ_α of an α -semiintegrable almost Grassmann structure are of dimension p . They are projected onto the original manifold M in the form of a submanifold V_α of the same dimension p , which, at any of its points, is tangent to the p -subspace ξ_α of the fiber bundle E_α . Through each point $x \in M$, there passes a $(q-1)$ -parameter family of submanifolds V_α .

Similarly, integral manifolds \tilde{V}_β of the distribution Δ_β of a β -semiintegrable almost Grassmann structure are of dimension q . They are projected onto the original manifold M in the form of a submanifold V_β of the same dimension q , which, at any of its points, is tangent to the q -subspace η_β of the fiber bundle E_β . Through each point $x \in M$, there passes a $(p-1)$ -parameter family of submanifolds V_β . If an almost Grassmann structure on M is integrable, then through each point $x \in M$, there pass a $(q-1)$ -parameter family of submanifolds V_α and a $(p-1)$ -parameter family of submanifolds V_β which were described above.

The Grassmann structure $G(m, n)$ is an integrable almost Grassmann structure $AG(m, n)$ since through any point x of the variety $\Omega(m, n)$, onto which the manifold $G(m, n)$ is mapped bijectively under the Grassmann mapping, there pass a $(q-1)$ -parameter family of p -dimensional plane generators (which are the submanifolds V_α) and a $(p-1)$ -parameter family of q -dimensional plane generators (which are the submanifolds V_β). In the projective space P^n , to submanifolds V_α there corresponds a family of m -dimensional subspaces belonging to a subspace of dimension $m+1$, and to submanifolds V_β there corresponds a family of m -dimensional subspaces passing through a subspace of dimension $m-1$.

Note that if the manifold M orientable, and we change its orientation, then an α -semiintegrable almost Grassmann structure will become β -integrable, and vice versa.

3. Consider a differentiable manifold M of dimension pq endowed with an almost Grassmann structure $AG(p-1, p+q-1)$. Suppose that $x \in M$, $T_x(M)$ is the tangent space of the manifold M at the point x , and that $\{e_i^\alpha\}$ is an adapted frame of the structure $AG(p-1, p+q-1)$. The decomposition of a vector $z \in T_x(M)$ with respect to this basis can be written in the form

$$z = \omega_\alpha^i(z) e_i^\alpha,$$

where ω_α^i are 1-forms making up the *co-frame* in the space $T_x(M)$. If $z = dx$ is the differential of a point $x \in M$, then the forms $\omega_\alpha^i(dx)$ are differential forms defined on a first-order frame bundle associated with the almost Grassmann structure. These forms constitute a completely integrable system of forms.

As was proved in [2], the form $\theta = (\omega_\alpha^i)$ and the forms arising under its prolongation satisfy the following structure equations:

$$\begin{aligned} d\omega_\alpha^i - \omega_\alpha^j \wedge \omega_j^i - \omega_\alpha^\beta \wedge \omega_\beta^i - \omega \wedge \omega_\alpha^i &= a_{\alpha j k}^{i \beta \gamma} \omega_\beta^j \wedge \omega_\gamma^k, \\ d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta &= \frac{q}{p+q} (\delta_\alpha^\beta \omega_\gamma^k \wedge \omega_k^\gamma - p \omega_\alpha^k \wedge \omega_k^\beta) + b_{\alpha k l}^{\beta \gamma \delta} \omega_\gamma^k \wedge \omega_\delta^l, \\ d\omega_j^i - \omega_j^k \wedge \omega_k^i &= \frac{p}{p+q} (\delta_j^i \omega_k^\gamma \wedge \omega_\gamma^k - q \omega_j^\gamma \wedge \omega_\gamma^i) + b_{j k l}^{i \gamma \delta} \omega_\gamma^k \wedge \omega_\delta^l, \\ d\omega &= \omega_i^\alpha \wedge \omega_\alpha^i, \\ d\omega_i^\alpha - \omega_i^\beta \wedge \omega_\beta^\alpha - \omega_i^j \wedge \omega_j^\alpha + \omega \wedge \omega_i^\alpha &= c_{i j k}^{\alpha \beta \gamma} \omega_\gamma^k \wedge \omega_\beta^j - a_{\gamma i j}^{k \alpha \beta} \omega_k^\gamma \wedge \omega_\beta^j, \end{aligned} \quad (1.4)$$

where the matrix 1-form $\theta = (\omega_\alpha^i)$ is defined in a first-order frame bundle, the form ω is a scalar form defined in a second-order frame bundle, $\omega_\alpha = (\omega_\beta^\alpha)$ and $\omega_\beta = (\omega_j^i)$ are the matrix 1-forms also defined in a second-order frame bundle. The forms ω_α^β and ω_i^j satisfy the conditions

$$\omega_\gamma^\gamma = 0, \quad \omega_k^k = 0. \quad (1.5)$$

In equations (1.4) the 2-form

$$\Theta_\alpha^i = a_{\alpha jk}^{i\beta\gamma} \omega_\beta^j \wedge \omega_\gamma^k \quad (1.6)$$

is the *torsion* 2-form of the $AG(p-1, p+q-1)$ -structure, and the forms

$$\Omega_\alpha^\beta = b_{\alpha kl}^{\beta\gamma\delta} \omega_\gamma^k \wedge \omega_\delta^l, \quad \Omega_j^i = b_{jkl}^{i\gamma\delta} \omega_\gamma^k \wedge \omega_\delta^l, \quad \Phi_i^\alpha = c_{ijk}^{\alpha\beta\gamma} \omega_\gamma^k \wedge \omega_\beta^j \quad (1.7)$$

are the *curvature* 2-forms of this structure.

Equations (1.4) define a connection similar to a *normal Cartan connection* constructed by É. Cartan [4] for conformal structures. Note that Dhooghe in [5] and [6] gave the structure equations of an almost Grassmann structure in the form close to our equations (1.4). His equations differ from equations (1.4) only by the additional term Ω_* in the right-hand side of the fourth equation.

Moreover, in his further considerations in [5] and [6] Dhooghe assumes that the torsion form Ω_α^i is identically equal to 0 not only for $p = q = 2$ but also for $p > 2$ and $q > 2$. This leads to the loss of generality. We did not make the above assumption.

In [2] (see also [1, §7.2]) we proved the following facts:

(a) The quantities $a = \{a_{\alpha jk}^{i\beta\gamma}\}$, defined by a second-order neighborhood, form a relative tensor of weight -1 and satisfy the following conditions:

$$a_{\alpha jk}^{i\beta\gamma} = -a_{\alpha kj}^{i\gamma\beta} \quad (1.8)$$

and

$$a_{\alpha jk}^{i\alpha\gamma} = 0, \quad a_{\alpha ik}^{i\beta\gamma} = 0. \quad (1.9)$$

The tensor $\{a_{\alpha jk}^{i\beta\gamma}\}$ is said to be the *first structure tensor*, or the *torsion tensor*, of an almost Grassmann manifold $AG(p-1, p+q-1)$.

(b) Let us set $b^1 = \{b_{jkl}^{i\gamma\delta}\}$, $b^2 = \{b_{\alpha kl}^{\beta\gamma\delta}\}$ and $b = (b^1, b^2)$. The quantities (a, b^1) and (a, b^2) form linear homogeneous objects. They represent two subobjects of the *second structure object* (a, b) of the almost Grassmann structure $AG(p-1, p+q-1)$. The components b^1 and b^2 satisfy the conditions

$$b_{\alpha kl}^{\beta\gamma\delta} = -b_{\alpha lk}^{\beta\delta\gamma}, \quad b_{jkl}^{i\gamma\delta} = -b_{jlk}^{i\delta\gamma}, \quad (1.10)$$

$$b_{\alpha kl}^{\alpha\gamma\delta} = 0, \quad b_{ikl}^{i\gamma\delta} = 0, \quad (1.11)$$

$$b_{\alpha kl}^{\gamma\alpha\delta} - b_{kil}^{i\gamma\delta} + b_{\alpha lk}^{\delta\alpha\gamma} - b_{lik}^{i\delta\gamma} = 0, \quad (1.12)$$

and the components of the tensor a satisfy the following differential equations:

$$2a_{\alpha[j|\beta]m}^{i[\beta]e} a_{\epsilon[kl]}^{m|\gamma\delta]} + \delta_{[j}^i b_{|\alpha[kl]}^{[\beta\gamma\delta]} - \delta_\alpha^{[\beta} b_{[jkl]}^{i|\gamma\delta]} + a_{\alpha[jkl]}^{i[\beta\gamma\delta]} = 0, \quad (1.13)$$

where $a_{\alpha j k l}^{i \beta \gamma \delta}$ are the Pfaffian derivatives of $a_{\alpha j k}^{i \beta \gamma}$ and the alternation is performed with respect to the vertical pairs of indices (β_j) and (γ_k) . Formulas (1.13) are analogues of the Bianchi equations in the theory of spaces with affine connection.

(c) For $p > 2$ and $q > 2$, the components of b^2 and b^1 are expressed in terms of the components of the tensor a and their Pfaffian derivatives.

This implies that if for $p > 2$, $q > 2$, the torsion tensor a of an almost Grassmann structure vanishes, then its curvature tensor b vanishes as well.

(d) Let us set $c = \{c_{i j k}^{\alpha \beta \gamma}\}$. Then $S = (a, b, c)$ forms a linear homogeneous object, which is called the *third structure object* of the almost Grassmann structure $AG(p-1, p+q-1)$. It is defined by a fourth-order differential neighborhood of $AG(p-1, p+q-1)$. Its subobject a is a relative tensor (the torsion tensor) defined by a second-order differential neighborhood of $AG(p-1, p+q-1)$, and the subobjects (a, b^1) , (a, b^2) , and (a, b) are defined by a third-order differential neighborhood of $AG(p-1, p+q-1)$.

The third structural object $S = (a, b, c)$ is the *complete geometric object* of the almost Grassmann structure $AG(p-1, p+q-1)$, since during the prolongation of structure equations (1.4) of $AG(p-1, p+q-1)$, all newly arising objects are expressed in terms of the components of the object S and their Pfaffian derivatives.

The components of c satisfy the conditions

$$c_{\alpha j k}^{i \beta \gamma} = -c_{\alpha k j}^{i \gamma \beta} \quad (1.14)$$

and

$$c_{[i j k]}^{[\alpha \beta \gamma]} = 0, \quad (1.15)$$

and the components of b satisfy the differential equations

$$\begin{aligned} b_{\alpha[k l m]}^{\beta[\gamma \delta \varepsilon]} - \frac{p q}{p+q} \delta_{\alpha}^{[\varepsilon} c_{[m k l]}^{|\beta| \gamma \delta]} - 2 b_{\alpha[k l s]}^{\beta[\gamma|\sigma} a_{\sigma|l m]}^{s|\delta \varepsilon]} &= 0, \\ b_{j[k l m]}^{i[\gamma \delta \varepsilon]} + \frac{p q}{p+q} \delta_{[m}^i c_{j|k l]}^{[\varepsilon \gamma \delta]} + 2 b_{j s[m}^{i \sigma[\varepsilon} a_{|\sigma|k l]}^{s|\gamma \delta]} &= 0, \end{aligned} \quad (1.16)$$

where $b_{j k l m}^{i \gamma \delta \varepsilon}$ and $b_{\alpha k l m}^{\beta \gamma \delta \varepsilon}$ are the Pfaffian derivatives of $b_{\alpha k l}^{\beta \gamma \delta}$ and $b_{\alpha k l}^{\beta \gamma \delta}$, respectively. In formulas (1.15) and (1.16) the alternation is carried out with respect to the vertical pairs of indices.

(e) If $p > 2$, then the components of c are expressed in terms of the components of the subobject (a, b^1) and their Pfaffian derivatives, and if $q > 2$, then the components of c are expressed in terms of the components of the subobject (a, b^2) and their Pfaffian derivatives. This implies that in this case the object (a, b) satisfies certain differential equations which are other analogues of the Bianchi equations in the theory of spaces with affine connection. These equations can be obtained if we substitute for the components of c in equations (1.16) their values.

(f) An almost Grassmann structure $AG(p-1, p+q-1)$ is said to be *locally Grassmann* (or *locally flat*) if it is locally equivalent to a Grassmann structure. For $p > 2$ and $q > 2$, an almost Grassmann structure $AG(p-1, p+q-1)$ is locally Grassmann if and only if its first structure tensor a vanishes. For $p = 2$, $q = 2$, the tensor $a_{\alpha j k}^{i \beta \gamma}$ vanishes, and the condition for the structure $AG(1, 3)$ to be locally Grassmann is the vanishing of the tensor b . The case $p = 2$, $q > 2$ (and $p > 2$, $q = 2$) will be considered below.

2. Semiintegrability of almost Grassmann structures

1. In this section we find geometric conditions for an almost Grassmann structure $AG(p-1, p+q-1)$ defined on a manifold M to be semiintegrable. The conditions are expressed in terms of the complete structure object S of the almost Grassmann structure $AG(p-1, p+q-1)$ and its subobjects S_α and S_β which will be defined in this section.

In what follows, we often encounter quantities satisfying the conditions similar to conditions (1.8) for the quantities $a_{\alpha j k}^{\beta \gamma}$. For calculations with quantities of this kind, the following lemma is very useful:

Lemma 2.1. *If a system of quantities $T_{\dots ij}^{\dots \alpha \beta}$ is skew-symmetric with respect to the pairs of indices $\binom{\alpha}{i}$ and $\binom{\beta}{j}$, namely satisfies the conditions*

$$T_{\dots ij}^{\dots \alpha \beta} = -T_{\dots ji}^{\dots \beta \alpha}, \quad (2.1)$$

then the following identities hold:

$$\begin{aligned} T_{\dots ij}^{[\alpha \beta]} &= T_{\dots (ij)}^{\alpha \beta}, & T_{\dots ij}^{(\alpha \beta)} &= T_{\dots [ij]}^{\alpha \beta}, \\ T_{\dots ij}^{(\alpha \beta)} &= -T_{\dots ji}^{(\alpha \beta)} = T_{\dots [ij]}^{(\alpha \beta)}, & T_{\dots [ij]}^{\alpha \beta} &= -T_{\dots [ij]}^{\beta \alpha} = T_{\dots [ij]}^{(\alpha \beta)}, \\ T_{\dots [ij]}^{[\alpha \beta]} &= 0, & T_{\dots (ij)}^{(\alpha \beta)} &= 0. \end{aligned} \quad (2.2)$$

In these relations the symmetrization and the alternation are carried separately over the lower indices and the upper indices. In addition the following decompositions take place:

$$T_{\dots ij}^{\alpha \beta} = T_{\dots ij}^{(\alpha \beta)} + T_{\dots (ij)}^{\alpha \beta}, \quad T_{\dots ij}^{\alpha \beta} = T_{\dots ij}^{[\alpha \beta]} + T_{\dots [ij]}^{\alpha \beta}. \quad (2.3)$$

Proof. All these identities can be proved by direct calculation with help of (2.1). \square

In addition, in the proof of the main theorem, we will use the following lemma:

Lemma 2.2. *The condition*

$$T_{[ijk]}^{[\alpha \beta \gamma]} = 0, \quad (2.4)$$

where the alternation is carried over three vertical pairs of indices, implies the condition

$$T_{(ijk)}^{[\alpha \beta \gamma]} = 0, \quad (2.5)$$

where the alternation and symmetrization are carried separately over the upper triple of indices and the lower triple of indices.

Proof. To prove this, one writes down 36 terms of $T_{(ijk)}^{[\alpha \beta \gamma]}$ and collects from them 6 groups of 6 terms to each of which the hypothesis (2.4) can be applied. \square

Next we will prove the following important result on the decomposition of the torsion tensor of an almost Grassmann structure $AG(p-1, p+q-1)$:

Theorem 2.3. *The torsion tensor $a = \{a_{\alpha jk}^{i\beta\gamma}\}$ of the almost Grassmann structure $AG(p-1, p+q-1)$ decomposes into two subtensors:*

$$a = a_\alpha + a_\beta, \quad (2.6)$$

where $a_\alpha = \{a_{\alpha(jk)}^{i\beta\gamma}\}$ and $a_\beta = \{a_{\alpha[jk]}^{i(\beta\gamma)}\}$.

Proof. Since the tensor $a_{\alpha jk}^{i\beta\gamma}$ is skew-symmetric with respect to the pairs of indices $\binom{\beta}{j}$ and $\binom{\gamma}{k}$, then, by Lemma 2.1, the decomposition (2.6) is equivalent to the obvious decomposition

$$a_{\alpha jk}^{i\beta\gamma} = a_{\alpha(jk)}^{i\beta\gamma} + a_{\alpha[jk]}^{i\beta\gamma}. \quad \square$$

Note that by Lemma 2.1, the subtensors a_α and a_β can be also represented in the form

$$a_\alpha = \{a_{\alpha jk}^{i[\beta\gamma]}\}, \quad a_\beta = \{a_{\alpha[jk]}^{i\beta\gamma}\}.$$

Note also that like the tensor a , its subtensors a_α and a_β are skew-symmetric with respect to the pairs of indices $\binom{\beta}{j}$ and $\binom{\gamma}{k}$:

$$a_{\alpha(jk)}^{i\beta\gamma} = -a_{\alpha(kj)}^{i\gamma\beta}, \quad a_{\alpha[jk]}^{i\beta\gamma} = -a_{\alpha[kj]}^{i\gamma\beta},$$

and they are also trace-free, since it follows from (1.9) that

$$a_{\alpha(jk)}^{i\alpha\gamma} = 0, \quad a_{\alpha ik}^{i[\beta\gamma]} = 0, \quad a_{\alpha ik}^{i(\beta\gamma)} = 0, \quad a_{\alpha[jk]}^{i\alpha\gamma} = 0. \quad (2.7)$$

Theorem 2.4. *If $p = 2$, then $a_\alpha = 0$, and if $q = 2$, then $a_\beta = 0$.*

Proof. Suppose that $p = 2$. Then $\alpha, \beta, \gamma = 1, 2$. Since, by definition and Lemma 2.1, the tensor a_α is skew-symmetric with respect to the indices β and γ , we have $a_{\alpha(jk)}^{i11} = a_{\alpha(jk)}^{i22} = 0$. But the first condition of (2.7) gives

$$a_{1(jk)}^{i11} + a_{2(jk)}^{i21} = 0, \quad a_{1(jk)}^{i12} + a_{2(jk)}^{i22} = 0.$$

It follows from these relations that $a_{2(jk)}^{i21} = a_{1(jk)}^{i12} = 0$; that is, all components of the tensor a_α vanish.

For the case $q = 2$, the proof is similar. \square

2. We introduce the following notation:

$$\begin{aligned} b_\alpha^1 &= \{b_{(jkl)}^{i\gamma\delta}\}, & b_\alpha^2 &= \{b_{\alpha kl}^{[\beta\gamma\delta]}\}, & c_\alpha &= \{c_{(ijk)}^{[\alpha\beta\gamma]}\}, \\ b_\beta^1 &= \{b_{[jkl]}^{i\gamma\delta}\}, & b_\beta^2 &= \{b_{\alpha kl}^{(\beta\gamma\delta)}\}, & c_\beta &= \{c_{[ijk]}^{(\alpha\beta\gamma)}\}. \end{aligned} \quad (2.8)$$

Now we give necessary and sufficient conditions for an almost Grassmann structure $AG(p-1, p+q-1)$ to be α -semiintegrable or β -semiintegrable.

Theorem 2.5. (i) *If $p > 2$ and $q \geq 2$, then for an almost Grassmann structure $AG(p-1, p+q-1)$ to be α -semiintegrable, it is necessary and sufficient that the conditions $a_\alpha = b_\alpha^1 = b_\alpha^2 = 0$ hold.*

(ii) *If $p \geq 2$ and $q > 2$, then for an almost Grassmann structure $AG(p-1, p+q-1)$ to be β -semiintegrable, it is necessary and sufficient that the conditions $a_\beta = b_\beta^1 = b_\beta^2 = 0$ hold.*

Proof. We prove part (i) of theorem. The proof of part (ii) is similar.

Suppose that θ_α , $\alpha = 1, \dots, p$, are basis forms of the integral submanifolds V_α , $\dim V_\alpha = p$, of the distribution Δ_α appearing in Definition 1.2. Then

$$\omega_\alpha^i = s^i \theta_\alpha, \quad \alpha = 1, \dots, p; \quad i = p+1, \dots, p+q, \quad (2.9)$$

where θ_α are basis forms on the submanifold V_α .

For the structure $AG(p-1, p+q-1)$ to be α -semiintegrable, it is necessary and sufficient that system (2.9) be completely integrable. Taking the exterior derivatives of equations (2.9) by means of structure equations (1.4), we find that

$$(ds^i + s^j \omega_j^i - s^i \omega) \wedge \theta_\alpha + s^i (d\theta_\alpha - \omega_\alpha^\beta \wedge \theta_\beta) = a_{\alpha jk}^{i\beta\gamma} s^j s^k \theta_\beta \wedge \theta_\gamma. \quad (2.10)$$

It follows from these equations that

$$d\theta_\alpha - \omega_\alpha^\beta \wedge \theta_\beta = \varphi_\alpha^\beta \wedge \theta_\beta, \quad (2.11)$$

where φ_α^β is an 1-form that is not expressed in terms of the basis forms θ_α .

For brevity, we set

$$\varphi^i = ds^i + s^j \omega_j^i - s^i \omega. \quad (2.12)$$

Then the exterior quadratic equation (2.10) takes the form

$$(\delta_\alpha^\beta \varphi^i + s^i \varphi_\alpha^\beta) \wedge \theta_\beta = a_{\alpha jk}^{i\beta\gamma} s^j s^k \theta_\beta \wedge \theta_\gamma. \quad (2.13)$$

From (2.13) it follows that for $\theta_\alpha = 0$, the 1-form $\delta_\alpha^\beta \varphi^i + s^i \varphi_\alpha^\beta$ vanishes:

$$\delta_\alpha^\beta \varphi^i(\delta) + s^i \varphi_\alpha^\beta(\delta) = 0. \quad (2.14)$$

Contracting equation (2.14) with respect to the indices α and β , we find that

$$\varphi^i = -s^i \varphi(\delta), \quad \varphi_\alpha^\beta = \delta_\alpha^\beta \varphi(\delta), \quad (2.15)$$

where we set $\varphi(\delta) = (1/p) \varphi_\gamma^\gamma(\delta)$.

It follows from (2.15) that on the subvariety V_α , the 1-forms φ^i and φ_α^β can be written as:

$$\varphi^i = -s^i \varphi + s^{i\beta} \theta_\beta, \quad \varphi_\alpha^\beta = \delta_\alpha^\beta \varphi + \hat{s}_\alpha^{\beta\gamma} \theta_\gamma. \quad (2.16)$$

Substituting these expressions into equations (2.11) and (2.12), we find that

$$d\theta_\alpha - \omega_\alpha^\beta \wedge \theta_\beta = \varphi \wedge \theta_\alpha + s_\alpha^{\beta\gamma} \theta_\gamma \wedge \theta_\beta, \quad (2.17)$$

where $s_\alpha^{\beta\gamma} = \hat{s}_\alpha^{[\beta\gamma]}$ and

$$ds^i + s^j \omega_j^i - s^i \omega = -s^i \varphi + s^{i\beta} \theta_\beta. \quad (2.18)$$

Substituting (2.17) and (2.18) into equation (2.10), we obtain

$$-s^i s_\alpha^{\beta\gamma} - \delta_\alpha^\beta s^{[i|\gamma]} = a_{\alpha jk}^{i\beta\gamma} s^j s^k. \quad (2.19)$$

Contracting equation (2.19) with respect to the indices α and β , we obtain

$$-2s^i s_\alpha^{\alpha\gamma} - p s^{i\gamma} + s^{i\gamma} = 0,$$

from which it follows that

$$s^{i\gamma} = s^i s^\gamma, \quad (2.20)$$

where we set $s^\gamma = -2s_\alpha^{\alpha\gamma}/(p-1)$. Substituting (2.20) into (2.19), we find that

$$s^i (\delta_\alpha^\gamma s^\beta - \delta_\alpha^\beta s^\gamma - 2s_\alpha^{\beta\gamma}) = 2a_{\alpha jk}^{i[\beta\gamma]} s^j s^k. \quad (2.21)$$

It follows that

$$\delta_\alpha^\gamma s^\beta - \delta_\alpha^\beta s^\gamma - 2s_\alpha^{\beta\gamma} = s_{\alpha j}^{\beta\gamma} s^j, \quad (2.22)$$

where $s_{\alpha j}^{\beta\gamma} = -s_{\alpha j}^{\gamma\beta}$. Substituting (2.22) into (2.21), we arrive at the equation

$$s_{\alpha(j}^{\beta\gamma} \delta_{k)}^i = a_{\alpha(jk)}^{i\beta\gamma}, \quad (2.23)$$

where the alternation sign in the right-hand side is dropped by Lemma 2.1.

Contracting (2.23) with respect to the indices i and j and taking into account of equations (1.9) and (1.10), we obtain

$$s_{\alpha k}^{\beta\gamma} = 0, \quad (2.24)$$

from which, by (2.23), it follows that

$$a_{\alpha(jk)}^{i\beta\gamma} = 0. \quad (2.25)$$

This proves that *if an almost Grassmann structure $AG(p-1, p+q-1)$ is α -semiintegrable, then its torsion tensor satisfies the condition (2.25), that is, $a_\alpha = 0$.*

Since, by Theorem 2.4, for $p = 2$ the subtensor $a_\alpha = 0$, condition (2.25) is identically satisfied. Hence, while proving sufficiency of this condition for α -semiintegrability, we must assume that $p > 2$.

Let us return to equations (2.17) and (2.18). Substitute into equation (2.18) the values $s^{i\beta}$ taken from (2.20) and set

$$\tilde{\varphi} = \varphi - s^\beta \theta_\beta. \quad (2.26)$$

In addition, by (2.24), relations (2.22) imply that $s_\alpha^{\beta\gamma} = \delta_\alpha^{[\gamma} s^{\beta]}$. Then equations (2.17) and (2.18) take the forms

$$d\theta_\alpha - (\omega_\alpha^\beta + \delta_\alpha^\beta \tilde{\varphi}) \wedge \theta_\beta = 0 \quad (2.27)$$

and

$$ds^i + s^j \omega_j^i - s^i (\omega - \tilde{\varphi}) = 0. \quad (2.28)$$

Taking the exterior derivatives of (2.28), we obtain the following exterior quadratic equations:

$$s^i \Phi + b_{jkl}^{i\gamma\delta} s^j s^k s^l \theta_\gamma \wedge \theta_\delta = 0, \quad (2.29)$$

where

$$\Phi = d\tilde{\varphi} - \frac{(p+1)q}{p+q} s^k \omega_k^\gamma \wedge \theta_\gamma.$$

Next, taking the exterior derivatives of (2.27), we find that

$$\Phi \wedge \theta_\alpha + b_{\alpha kl}^{\beta\gamma\delta} s^k s^l \theta_\beta \wedge \theta_\gamma \wedge \theta_\delta = 0. \quad (2.30)$$

Equations (2.29) show that the 2-form Φ can be written as

$$\Phi = A_{kl}^{\gamma\delta} s^k s^l \theta_\gamma \wedge \theta_\delta, \quad (2.31)$$

where the coefficients $A_{kl}^{\gamma\delta}$ are symmetric with respect to the lower indices and skew-symmetric with respect to the upper indices. Substituting this value of the form Φ into equations (2.29) and (2.30), we arrive at the conditions

$$b_{(jkl)}^{i[\gamma\delta]} + \delta_{(j}^i A_{kl)}^{\gamma\delta} = 0 \quad (2.32)$$

and

$$b_{\alpha(kl)}^{[\beta\gamma\delta]} + \delta_\alpha^{[\beta} A_{kl]}^{\gamma\delta]} = 0. \quad (2.33)$$

Contracting equation (2.32) with respect to the indices i and j and equation (2.33) with respect to the indices α and β , we find that

$$2(q+2)A_{kl}^{\gamma\delta} + b_{kli}^{i\gamma\delta} + b_{kil}^{i\gamma\delta} + b_{lik}^{i\gamma\delta} + b_{lki}^{i\gamma\delta} = 0 \quad (2.34)$$

and

$$2(p-2)A_{kl}^{\gamma\delta} + b_{\alpha kl}^{\gamma\delta\alpha} + b_{\alpha lk}^{\gamma\delta\alpha} + b_{\alpha kl}^{\delta\alpha\gamma} + b_{\alpha lk}^{\delta\alpha\gamma} = 0. \quad (2.35)$$

Note that for $p = 2$ equation (2.33) becomes an identity, and we will not obtain equation (2.35).

If we add equations (2.34) and (2.35) and apply condition (1.12), we find that

$$A_{kl}^{\gamma\delta} = 0. \quad (2.36)$$

As a result, equations (2.32) and (2.33) take the form

$$b_{(jkl)}^{i[\gamma\delta]} = 0, \quad b_{\alpha(kl)}^{[\beta\gamma\delta]} = 0. \quad (2.37)$$

By Lemma 2.1, conditions (2.37) are equivalent to the conditions

$$b_{(jkl)}^{i\gamma\delta} = 0, \quad b_{\alpha kl}^{[\beta\gamma\delta]} = 0. \quad (2.38)$$

It follows from equations (2.36) and (2.31) that

$$d\tilde{\varphi} = \frac{(p+1)q}{p+q} s^k \omega_k^\gamma \wedge \theta_\gamma. \quad (2.39)$$

Finally, taking the exterior derivatives of equations (2.39) and applying (2.27), (2.28), and (1.4), we obtain the condition

$$c_{(ijk)}^{[\alpha\beta\gamma]} = 0. \quad (2.40)$$

These equations will not be trivial only if $p > 2$. But, by Lemma 2.2, conditions (2.40) follow from integrability conditions (1.15).

Thus the system of Pfaffian equations (2.9), defining integral submanifolds of an α -semiintegrable almost Grassmann structure, together with Pfaffian equations (2.28) and (2.39) following from (2.9), is completely integrable if and only if conditions (2.25) and (2.38) are satisfied. This proves part (i).

As we noted in the beginning, the proof of part (ii) is similar. We note only that the equations of integral submanifolds V_β , $\dim V_\beta = q$, of the distribution Δ_β appearing in Definition 1.6 can be written in the form

$$\omega_\alpha^i = s_\alpha \theta^i, \quad \alpha = 1, \dots, p; \quad i = p + 1, \dots, p + q, \quad (2.41)$$

where the 1-forms θ^i are linearly independent on the submanifold V_β . \square

It follows from our previous considerations and (2.8) that

1. for $p = 2$ we have $b_\alpha^2 = 0$ and $c_\alpha = 0$;
2. for $q = 2$ we have $b_\beta^1 = 0$ and $c_\beta = 0$;
3. for $p > 2$ we have $c_\alpha = 0$;
4. for $q > 2$ we have $c_\beta = 0$.

The last two results follow from conditions (1.15) and Lemma 2.2. These results combined with differential equations which the components of b^1 and b^2 satisfy imply that the tensors a_α, a_β and the quantities $b_\alpha^1, b_\alpha^2, b_\beta^1, b_\beta^2$ form the following geometric objects:

$$\begin{aligned} (a_\alpha, b_\alpha^1), \quad (a_\alpha, b_\alpha^2), \quad S_\alpha &= (a_\alpha, b_\alpha^1, b_\alpha^2), \\ (a_\beta, b_\beta^1), \quad (a_\beta, b_\beta^2), \quad S_\beta &= (a_\beta, b_\beta^1, b_\beta^2), \end{aligned}$$

which are subobjects of the second structural object and the complete structural object of the almost Grassmann structure.

The following theorem gives the conditions of α - and β -semiintegrability in the cases when $p = 2$ or $q = 2$.

Theorem 2.6. (i) If $p = 2$, then the structure subobject S_α consists only of the tensor b_α^1 , and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(1, q + 1)$ to be α -semiintegrable.

(ii) If $q = 2$, then the structure subobject S_β consists only of the tensor b_β^2 , and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(p - 1, p + 1)$ (which is equivalent to the structure $AG(1, p + 1)$) to be β -semiintegrable.

(iii) If $p = q = 2$, then the complete structural object S consists only of the tensors b_α^1 and b_β^2 , and the vanishing of one of these tensors is necessary and sufficient for the almost Grassmann structure $AG(1, 3)$ to be α - or β -semiintegrable, respectively.

Proof. We will prove part (i). As we have already seen, for $p = 2$, the tensor a_α as well as the quantities b_α^2 and c_α vanish ($a_\alpha = b_\alpha^2 = c_\alpha = 0$), and the object b_α^1 becomes a tensor. Thus the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(1, q + 1)$ to be α -semiintegrable. The proof of part (ii) is similar. Part (iii) combines the results of (i) and (ii). \square

We will make two more remarks:

1. The tensors b_α^1 and b_β^2 are defined by a third-order differential neighborhood of the almost Grassmann structure.

2. For $p = q = 2$, as was indicated earlier (see Subsection 1.1), the almost Grassmann structure $AG(1, 3)$ is equivalent to the conformal $CO(2, 2)$ -structure. Thus we have the following decomposition of its complete structural object: $S = b_\alpha^1 + b_\beta^2$ (see [1, §5.1]).

Note also that in [5] and [6] the author assumed that an almost Grassmann structure $AG(p-1, p+q-1)$ is semiintegrable if and only if one of two curvature forms Ω_i^j or Ω_α^β vanishes. However, our previous considerations as well as formula (3.18) below show that this will be the case only if $p = q = 2$.

3. Existence of semiintegrable almost Grassmann structures

1. We will prove the existence of four-dimensional and $(2p)$ - and $(2q)$ -dimensional semiintegrable almost Grassmann structures, where $p, q > 2$, by constructing examples of such structures.

In order to prove that a certain almost Grassmann structure is α - or β -semiintegrable, we will use two methods.

The *first method* is to check whether conditions of α - or β -semiintegrability outlined in Theorems 2.5 and 2.6 are satisfied.

When we apply this method, we will need the following lemma:

Lemma 3.1. *If the forms ω , ω_α^β , and ω_j^i , occurring in equations (1.4) are principal forms, that is,*

$$\omega, \omega_\alpha^\beta, \omega_j^i \equiv 0 \pmod{\omega_\alpha^i},$$

then the form ω can be reduced to 0, and the forms ω_i^α , $\alpha = 1, 2$, $i = 3, 4$, become principal forms with a symmetric matrix of coefficients.

Proof. Suppose that ω is expressed in terms of the basis forms ω_α^i :

$$\omega = a\omega_1^3 + b\omega_2^3 + c\omega_1^4 + e\omega_2^4. \quad (3.1)$$

If we take exterior derivative of this equation, apply Cartan's lemma to the obtained exterior quadratic equation, and set $\omega_\alpha^i = 0$, we will obtain the following Pfaffian equations:

$$\begin{aligned} \delta a + \pi_3^1 &= 0, & \delta b + \pi_3^2 &= 0, \\ \delta c + \pi_4^1 &= 0, & \delta e + \pi_4^2 &= 0, \end{aligned}$$

where δ is the symbol of differentiation with respect to fiber parameters, and $\pi_i^\alpha = \omega_i^\alpha|_{\omega_\alpha^i=0}$.

This implies that the quantities a, b, c and e can be reduced to 0, $a = b = c = e = 0$. Let us show, for example, that a can be reduced to 0. In fact, since the forms $\pi_3^1, \pi_3^2, \pi_4^1$, and π_4^2 are linearly independent, we can set $\pi_3^2 = \pi_4^1 = \pi_4^2 = 0$ preserving $\pi_3^1 \neq 0$. Since now π_3^1 depends on one fiber parameter, for an appropriate choice of this parameter we have $\pi_3^1 = \delta t$. By integrating the equation $\delta a + \delta t = 0$, we obtain $a = -t + C$, where C is a constant. For any value of C , we can take $t = C$, and as a result, we will get $a = 0$.

Conversely, if the fiber parameters are chosen in such a way that $a = 0$, we get from the above differential equations that $\pi_3^1 = 0$.

The remaining reductions $b = c = e = 0$ can be proved in a similar manner. This gives $\pi_3^2 = \pi_4^1 = \pi_4^2 = 0$. After these reductions equation (3.1) becomes

$$\omega = 0. \quad (3.2)$$

Now equations (1.4) and (3.2) imply that

$$\omega_i^\alpha \wedge \omega_\alpha^i = 0.$$

Applying Cartan's lemma to this exterior quadratic equation, we obtain that the forms ω_i^α become principal forms, and they are expressed in terms of the basis forms ω_α^i as follows:

$$\begin{aligned} \omega_3^1 &= A_1\omega_1^3 + A_2\omega_2^3 + A_3\omega_1^4 + A_4\omega_2^4, \\ \omega_3^2 &= A_2\omega_1^3 + B_2\omega_2^3 + B_3\omega_1^4 + B_4\omega_2^4, \\ \omega_4^1 &= A_3\omega_1^3 + B_3\omega_2^3 + C_3\omega_1^4 + C_4\omega_2^4, \\ \omega_4^2 &= A_4\omega_1^3 + B_4\omega_2^3 + C_4\omega_1^4 + E_4\omega_2^4. \quad \square \end{aligned} \quad (3.3)$$

Note that the conditions for fiber forms in Lemma 3.1 mean the group G of admissible transformations of first-order frames is reduced to the identity group, $G = \{e\}$, and instead of a *fibration* of first-order frames associated with an $AG(1, 3)$ -structure, we have a *distribution* of such frames. In other words, now with any point of a manifold M^4 on which the $AG(1, 3)$ -structure is given only *one frame* is associated.

The normalization (3.2) and relations (3.3) implied by this normalization means that the subgroup $T(4)$ of translations contained in the prolonged group G' (see [1, p. 274]) is also reduced to the identity group, and $G' = G = \{e\}$. Furthermore, only one second-order frame is associated with any point $x \in M^4$. Moreover, the normalization (3.2)–(3.3) singles out a pseudo-Riemannian metric g of signature $(2, 2)$ that is concordant with the almost Grassmann structure $AG(1, 3)$ given on the manifold M^4 .

When we use the first method, we also need to use relations among the components of the tensor $b = \{b^1, b^2\}$ that follow from equations (1.12) and (1.13). There are 16 equations (1.12) and 256 equations (1.13). In order to get the relations along components of the tensor b , first we prove the following two lemmas which list 10 independent relations among relations (1.12) and 16 independent relations among relations (1.13).

Lemma 3.2. *For $p = q = 2$, equations (1.12) take the form:*

$$\begin{aligned} b_{233}^{121} - b_{343}^{411} &= 0, & b_{133}^{212} - b_{343}^{422} &= 0, \\ b_{244}^{121} - b_{434}^{311} &= 0, & b_{144}^{212} - b_{434}^{322} &= 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} 2b_{133}^{112} - b_{343}^{412} - b_{343}^{421} &= 0, & 2b_{144}^{112} - b_{434}^{312} - b_{434}^{321} &= 0, \\ 2b_{443}^{411} - b_{234}^{121} - b_{243}^{121} &= 0, & 2b_{334}^{322} - b_{143}^{212} - b_{134}^{212} &= 0, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} 2b_{134}^{112} + b_{234}^{122} - 2b_{334}^{312} - b_{344}^{412} + b_{143}^{211} - b_{433}^{321} &= 0, \\ b_{134}^{211} + 2b_{234}^{221} - 2b_{334}^{321} - b_{344}^{421} + b_{243}^{122} - b_{433}^{312} &= 0. \end{aligned} \quad (3.6)$$

Proof. The proof is straightforward. Equations (3.4) are obtained from (1.12) taking $\gamma = \delta = 1, k = l = 3$; $\gamma = \delta = 2, k = l = 3$; $\gamma = \delta = 1, k = l = 4$, and $\gamma = \delta = 2, k = l = 4$, respectively. Equations (3.5) are obtained from (1.12) taking $\gamma = 1, \delta = 2, k = l = 3$; $\gamma = 1, \delta = 2, k = l = 4$; $\gamma = \delta = 1, k = 3, l = 4$, and $\gamma = \delta = 2, k = 3, l = 4$, respectively. Finally, equations (3.6) are obtained from (1.12) taking $\gamma = 1, \delta = 2, k = 3, l = 4$ and $\gamma = 2, \delta = 1, k = 3, l = 4$, respectively. The remaining equations (1.12) do not give new conditions. Note that while obtaining (3.4)–(3.6), one should use conditions (1.10) and (1.11).

The remaining relations (1.12) are satisfied identically or lead to the same relations (3.4)–(3.6). \square

Lemma 3.3. For $p = q = 2$, equations (1.13) take the form:

$$\begin{aligned} b_{233}^{121} - b_{334}^{411} &= 0, & b_{133}^{212} - b_{334}^{422} &= 0, \\ b_{244}^{121} - b_{443}^{311} &= 0, & b_{144}^{212} - b_{443}^{322} &= 0, \end{aligned} \quad (3.7)$$

$$\begin{aligned} b_{133}^{121} &= b_{343}^{412} + b_{433}^{421}, & b_{233}^{212} &= b_{343}^{421} + b_{433}^{412}, \\ b_{144}^{121} &= b_{434}^{312} + b_{344}^{321}, & b_{244}^{212} &= b_{343}^{421} + b_{434}^{321}, \\ b_{334}^{311} &= b_{234}^{211} + b_{243}^{112}, & b_{443}^{411} &= b_{234}^{112} + b_{243}^{211}, \\ b_{334}^{322} &= b_{134}^{122} + b_{143}^{221}, & b_{443}^{422} &= b_{143}^{122} + b_{134}^{221}, \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} b_{334}^{312} + b_{433}^{321} &= b_{234}^{212} + b_{243}^{122}, & b_{143}^{211} + b_{134}^{112} &= b_{344}^{421} + b_{434}^{412}, \\ b_{334}^{321} + b_{433}^{312} &= b_{134}^{121} + b_{143}^{211}, & b_{234}^{221} + b_{243}^{122} &= b_{344}^{412} + b_{434}^{421}. \end{aligned} \quad (3.9)$$

Proof. The proof is also straightforward. Equations (3.7) are obtained from (1.13) $\alpha = \gamma = 2, \beta = \delta = 1, i = j = 4, k = l = 3$; $\alpha = \gamma = 1, \beta = \delta = 2, i = j = 4, k = l = 3$; $\alpha = \gamma = 2, \beta = \delta = 1, i = j = 3, k = l = 4$; and $\alpha = \gamma = 1, \beta = \delta = 2, i = j = 3, k = l = 4$, respectively.

Equations (3.8) are obtained from (1.13) taking $\alpha = \beta = \delta = 1, \gamma = 2, i = j = 4, k = l = 3$; $\alpha = \beta = \gamma = 2, \delta = 1, i = k = 4, j = l = 3$; $\alpha = \beta = \delta = 1, \gamma = 2, i = j = 3, k = l = 4$; $\alpha = \beta = \gamma = 2, \delta = 1, i = k = 4, j = l = 3$; $\alpha = \beta = 2, \gamma = \delta = 1, i = j = k = 3, l = 4$; $\alpha = \beta = 2, \gamma = \delta = 1, i = j = k = 4, l = 3$; $\alpha = \beta = 1, \gamma = \delta = 2, i = j = k = 3, l = 4$; and $\alpha = \beta = 1, \delta = \gamma = 2, i = j = k = 4, l = 3$, respectively.

Finally, equations (3.9) are obtained from (1.13) taking $\alpha = \beta = \delta = 2, \gamma = 1, i = j = k = 3, l = 4$; $\alpha = \beta = \gamma = 1, \delta = 2, i = k = l = 4, k = 3$; $\alpha = \beta = \delta = 1, \gamma = 2, i = j = k = 3, l = 4$; and $\alpha = \beta = \gamma = 2, \delta = 1, i = j = l = 4, k = 3$, respectively.

The remaining relations (1.13) are satisfied identically or lead to the same relations (3.7)–(3.9). \square

Theorem 3.4. *The components of the tensor b of an almost Grassmann structure $AG(1, 3)$ satisfy the following conditions:*

$$b_{233}^{121} = b_{334}^{411} = b_{133}^{212} = b_{334}^{422} = b_{244}^{121} = b_{443}^{311} = b_{144}^{212} = b_{443}^{322} = 0, \quad (3.10)$$

$$\begin{aligned} b_{133}^{112} &= 0, & b_{343}^{412} &= b_{433}^{412} = b_{334}^{412}, \\ b_{144}^{112} &= 0, & b_{443}^{321} &= b_{344}^{321} = b_{434}^{321}, \\ b_{443}^{411} &= 0, & b_{243}^{211} &= b_{243}^{122} = b_{243}^{121}, \\ b_{334}^{322} &= 0, & b_{134}^{122} &= b_{134}^{221} = b_{134}^{212}, \end{aligned} \quad (3.11)$$

and

$$b_{134}^{112} = b_{134}^{211} = b_{134}^{121} = b_{243}^{122}, \quad b_{433}^{321} = b_{334}^{321} = b_{343}^{321} = b_{344}^{412}. \quad (3.12)$$

Proof. To prove this theorem, we combine results of Lemmas 3.2 and 3.3. For example, the first two relations (3.10) are obtained by comparing the first equations of (3.4) and (3.7), and the first two equations (3.11) are obtained by comparing the first equation of (3.5) with the sum of first two equations of (3.8).

To prove relations (3.12), we first note that it follows from (3.9) and (3.6) that

$$\begin{aligned} 2b_{334}^{312} + b_{344}^{412} + b_{433}^{321} &= 0, & 2b_{134}^{112} + b_{234}^{122} + b_{143}^{211} &= 0, \\ 2b_{334}^{321} + b_{344}^{421} + b_{433}^{312} &= 0, & 2b_{234}^{221} + b_{143}^{122} + b_{134}^{211} &= 0. \end{aligned} \quad (3.13)$$

In fact, by adding the first two equations of (3.9) we find that

$$2b_{334}^{312} + b_{434}^{412} + b_{433}^{321} = -(2b_{134}^{112} + b_{234}^{122} + b_{143}^{211}).$$

Comparing this equation with the first equation of (3.6), we arrived to the first two equations of (3.13). The remaining equations of (3.13) are obtained from the last two equations of (3.9) and the second equation of (3.6).

If we add the equations of the first and the second column of (3.13), we easily find that

$$b_{334}^{321} = b_{343}^{321}, \quad b_{134}^{121} = b_{134}^{112}. \quad (3.14)$$

Next adding equations of the first row and the first column of (3.9) and taking into account of relations (3.13), we obtain

$$b_{134}^{211} = b_{243}^{122}, \quad b_{433}^{321} = b_{344}^{412}. \quad (3.15)$$

Equations (3.15) and the first two equations (3.13) imply that

$$b_{134}^{121} = b_{243}^{122}, \quad b_{343}^{321} = b_{344}^{412}. \quad (3.16)$$

Equations (3.15) and (3.16) gives relations (3.12). \square

It is easy to see from relations (3.10)–(3.12) that there are only 10 independent components of the curvature tensor b of an almost Grassmann structure $AG(1, 3)$, and that we were able to

obtain relations (3.10)–(3.12) only because for $p = q = 2$ conditions (1.13) become conditions for the components of the curvature tensor b of an almost Grassmann structure $AG(1, 3)$.

We make the following remark on a four-dimensional almost Grassmann structure $AG(1, 3)$. Since an $AG(1, 3)$ -structure is equivalent to a $CO(2, 2)$ -structure, we can calculate the conformal curvature tensor $C = C_\alpha + C_\beta$, where $C_\alpha = \{a_0, a_1, a_2, a_3, a_4\}$ and $C_\beta = \{b_0, b_1, b_2, b_3, b_4\}$ (see [1, Section 5.1]) of the latter.

We impose the following relations between the basis forms $\omega^1, \omega^2, \omega^3, \omega^4$ of a $CO(2, 2)$ -structure and the basis forms $\omega_1^3, \omega_2^3, \omega_1^4, \omega_2^4$ of an equivalent $AG(1, 3)$ -structure:

$$\begin{aligned}\omega^1 &= \frac{1}{\sqrt{2}} \omega_1^3, & \omega^2 &= \frac{1}{\sqrt{2}} \omega_2^3, \\ \omega^3 &= \frac{1}{\sqrt{2}} \omega_1^4, & \omega^4 &= \frac{1}{\sqrt{2}} \omega_2^4.\end{aligned}\tag{3.17}$$

The factor $1/\sqrt{2}$ can be explained by the fact that the metric of a $CO(2, 2)$ -structure usually is written in the form $g = 2(\omega^1\omega^4 - \omega^2\omega^3)$ and that the metric of an equivalent $AG(1, 3)$ -structure has the form $g = \omega_1^3\omega_2^4 - \omega_2^3\omega_1^4$ (see [1, Ch. 5 and 7]), and relations (3.13) make these metrics equal.

The calculations involving the apparatus developed in [1, Ch. 5] and the formulas (1.4)–(1.5), (3.4)–(3.5), give the following relations between the independent 10 components of the curvature tensor b of the structure $AG(1, 3)$ (see Theorem 3.4) and the independent 10 components of the tensor C of the equivalent pseudoconformal structure $CO(2, 2)$:

$$\begin{aligned}a_0 &= b_{333}^{412}, & b_0 &= b_{243}^{111}, \\ a_1 &= b_{433}^{412} = b_{334}^{412} = b_{343}^{412}, & b_1 &= b_{243}^{211} = b_{243}^{112} = b_{243}^{121}, \\ a_2 &= b_{433}^{321} = b_{334}^{321} = b_{343}^{321} = b_{344}^{412}, & b_2 &= b_{134}^{112} = b_{134}^{211} = b_{134}^{121} = b_{243}^{122}, \\ a_3 &= b_{344}^{321} = b_{434}^{321} = b_{443}^{321}, & b_3 &= b_{134}^{122} = b_{134}^{212} = b_{134}^{221}, \\ a_4 &= b_{444}^{321}, & b_4 &= b_{134}^{222}.\end{aligned}\tag{3.18}$$

The conditions $C_\alpha = 0$, $C_\beta = 0$ and $C = 0$ are necessary and sufficient for a $CO(2, 2)$ -structure (and consequently for an equivalent $AG(1, 3)$ -structure) to be α -semiintegrable, β -semiintegrable and locally flat, respectively. Moreover, if we find components of C_α and C_β , then by investigating the roots of polynomials

$$C_\alpha(\lambda) = a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4$$

and

$$C_\beta(\mu) = b_0\lambda^4 - 4b_1\lambda^3 + 6b_2\lambda^2 - 4b_3\lambda + b_4$$

we can make some additional conclusions on integrability of the distributions $\Delta_\alpha(\lambda)$ and $\Delta_\beta(\mu)$ of the fiber bundles E_α and E_β associated with a $CO(2, 2)$ -structure (see [1, Section 5.4]).

Note that equations (3.18) show that for a four-dimensional almost Grassmann structure $AG(1, 3)$ and for an equivalent pseudoconformal structure $CO(2, 2)$ we have

(i) The condition of α -semiintegrability $C_\alpha = 0$ is equivalent to the condition $b_{jkl}^{i\beta\gamma} = 0$, that is, $\Omega_i^j = 0$.

(ii) The condition of β -semiintegrability $C_\beta = 0$ is equivalent to the condition $b_{\alpha jk}^{\beta\gamma\delta} = 0$, that is, $\Omega_\alpha^\beta = 0$.

(iii) The condition of local flatness $C = 0$ is equivalent to the vanishing of the tensor $b = \{b_\alpha^1, b_\beta^2\}$ (cf. the end of Section 1).

Note that only for a four-dimensional almost Grassmann structure $AG(1, 3)$ the conditions of α - and β -semiintegrability are equivalent to $\Omega_i^j = 0$ and $\Omega_\alpha^\beta = 0$, respectively: only for such structures the components of the tensor b satisfy relations (3.10)–(3.12) by means of which the conditions $b_{\alpha kl}^{[\beta\gamma\delta]} = 0$ and $b_{[jkl]}^{i\gamma\delta} = 0$ of α - and β -semiintegrability imply the vanishing of the components $b_{\alpha kl}^{\beta\gamma\delta}$ and $b_{jkl}^{i\gamma\delta}$ themselves.

2. The *second method* is the method of direct integration of equations (2.9) and (2.41) defining integral submanifolds of distributions Δ_α and Δ_β of semiintegrable almost Grassmann structures. Let us describe this method in more detail.

In the proof of Theorem 2.5 we wrote the equations of the submanifolds V_α and V_β in the form (2.9) and (2.41), respectively. Note that $\dim V_\alpha = p$ and $\dim V_\beta = q$. An almost Grassmann structure is α -semiintegrable (or β -semiintegrable) if and only if the system of equations (2.9) (respectively, (2.41)) is completely integrable.

We write the matrix of basis forms of the almost Grassmann manifold $AG(p-1, p+q-1)$ in more detail:

$$(\omega_\alpha^i) = \begin{pmatrix} \omega_1^{p+1} & \omega_2^{p+1} & \dots & \omega_p^{p+1} \\ \omega_1^{p+2} & \omega_2^{p+2} & \dots & \omega_p^{p+2} \\ \dots & \dots & \dots & \dots \\ \omega_1^{p+q} & \omega_2^{p+q} & \dots & \omega_p^{p+q} \end{pmatrix}. \quad (3.19)$$

Here $\alpha = 1, \dots, p$ is the column number, and $i = p+1, \dots, p+q$ is the row number.

The condition (2.9) of α -integrability means that on integral submanifolds V_α of the distribution Δ_α (see Definition 1.2) the rows of the matrix (ω_α^i) are proportional, and the entries of every nonzero row are basis forms on V_α .

The condition (2.41) of β -integrability means that on integral submanifolds V_β of the distribution Δ_β (see Definition 1.2) the columns of the matrix (ω_α^i) are proportional, and the entries of every nonzero row are basis forms on V_β .

For $p = q = 2$, equations (2.9) of submanifolds V_α can be written in the form

$$\lambda\omega_1^3 + \omega_1^4 = 0, \quad \lambda\omega_2^3 + \omega_2^4 = 0, \quad (3.20)$$

where $\lambda = -s^4/s^3$ and $\omega_1^3 \wedge \omega_2^3 \neq 0$, and equations (2.41) of submanifolds V_β can be written in the form

$$\mu\omega_2^3 + \omega_1^3 = 0, \quad \mu\omega_2^4 + \omega_1^4 = 0, \quad (3.21)$$

where $\mu = -s_1/s_2$ and $\omega_2^3 \wedge \omega_2^4 \neq 0$.

If $p > 2$ or $q > 2$, the systems (3.20) and (3.21) have different forms. For example, let us consider the case $p = 2$ and $q = 3$. In this case equations (2.9) can be written as follows:

$$\begin{aligned} \lambda_1\omega_1^3 + \omega_1^4 &= 0, & \lambda_2\omega_1^3 + \omega_1^5 &= 0, \\ \lambda_1\omega_2^3 + \omega_2^4 &= 0, & \lambda_2\omega_2^3 + \omega_2^5 &= 0, \end{aligned} \quad (3.22)$$

where $\lambda_1 = -s^4/s^3$ and $\lambda_2 = -s^5/s^3$ and $\omega_1^3 \wedge \omega_2^3 \neq 0$, and equations (2.41) take the form:

$$\mu\omega_2^3 + \omega_1^3 = 0, \quad \mu\omega_2^4 + \omega_1^4 = 0, \quad \mu\omega_2^5 + \omega_1^5 = 0, \quad (3.23)$$

where $\mu = -s_1/s_2$ and $\omega_2^3 \wedge \omega_2^4 \wedge \omega_2^5 \neq 0$.

For these cases to prove that an almost Grassmann structure is α -semiintegrable (respectively β -semiintegrable) we must prove that the system (3.20) or (3.22) (respectively the system (3.21) or (3.23)) is completely integrable. If it is possible, we integrate these systems and find λ or λ_1 and λ_2 (resp. μ) and closed form equations of submanifolds V_α (resp. V_β).

3. We next construct examples of semiintegrable and integrable almost Grassmann structures $AG(1, 3)$. To prove that they are semiintegrable, we will apply one of two methods indicated above.

Example 3.5. Suppose that x, y, u , and v are coordinates in M^4 , and that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned} \omega_1^3 &= dx + f(u) dy, & \omega_2^3 &= dy, \\ \omega_1^4 &= du, & \omega_2^4 &= dv. \end{aligned} \quad (3.24)$$

Taking exterior derivatives of equations (3.22) by means of (1.4), (1.5) and (3.22), we arrive at the following exterior quadratic equations:

$$\begin{aligned} (\omega + \omega_1^1 - \omega_3^3) \wedge \omega_1^3 + \omega_1^2 \wedge \omega_2^3 + \omega_1^4 \wedge \omega_4^3 &= f'(u) \omega_1^4 \wedge \omega_2^4, \\ (\omega - \omega_1^1 - \omega_3^3) \wedge \omega_2^3 + \omega_2^1 \wedge \omega_1^3 + \omega_2^4 \wedge \omega_4^3 &= 0, \\ (\omega + \omega_1^1 + \omega_3^3) \wedge \omega_1^4 + \omega_1^2 \wedge \omega_2^4 + \omega_1^3 \wedge \omega_3^4 &= 0, \\ (\omega - \omega_1^1 + \omega_3^3) \wedge \omega_2^4 + \omega_2^1 \wedge \omega_1^4 + \omega_2^3 \wedge \omega_3^4 &= 0. \end{aligned} \quad (3.25)$$

First, equations (3.25) prove that the form ω is a principal form. Thus, by Lemma 3.1 we have equations (3.2) and (3.3). Second, it follows from (3.25), that the forms $\omega_1^2, \omega_2^1, \omega_3^4, \omega_4^3, \omega_1^1 + \omega_3^3$ and $\omega_1^1 - \omega_3^3$ are principal forms. We will write their expressions as follows:

$$\begin{aligned} \omega_1^2 &= \alpha_1 \omega_1^3 + \alpha_2 \omega_2^3 + \alpha_3 \omega_1^4 + \alpha_4 \omega_2^4, \\ \omega_2^1 &= \beta_1 \omega_1^3 + \beta_2 \omega_2^3 + \beta_3 \omega_1^4 + \beta_4 \omega_2^4, \\ \omega_3^4 &= \gamma_1 \omega_1^3 + \gamma_2 \omega_2^3 + \gamma_3 \omega_1^4 + \gamma_4 \omega_2^4, \\ \omega_4^3 &= \delta_1 \omega_1^3 + \delta_2 \omega_2^3 + \delta_3 \omega_1^4 + \delta_4 \omega_2^4, \\ \omega_1^1 + \omega_3^3 &= \sigma_1 \omega_1^3 + \sigma_2 \omega_2^3 + \sigma_3 \omega_1^4 + \sigma_4 \omega_2^4, \\ \omega_1^1 - \omega_3^3 &= \tau_1 \omega_1^3 + \tau_2 \omega_2^3 + \tau_3 \omega_1^4 + \tau_4 \omega_2^4. \end{aligned} \quad (3.26)$$

Substituting (3.2) and (3.26) into equations (3.25) and equating coefficients in the independent exterior forms $\omega_\alpha^i \wedge \omega_\beta^j$ to 0, we obtain: $\alpha_1 = \alpha_2 = \alpha_4 = 0$; $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$;

$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$; $\delta_1 = \delta_3 = \delta_4 = 0$; $\sigma_1 = \sigma_2 = \sigma_4 = 0$; $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, and $\alpha_3 = \delta_2 = \sigma_4 = \frac{1}{2}f'(u)$. As a result, equations (3.26) become

$$\begin{aligned}\omega_1^2 &= \frac{1}{2}f'(u)\omega_1^4, & \omega_2^1 &= 0, \\ \omega_4^3 &= \frac{1}{2}f'(u)\omega_2^3, & \omega_3^4 &= 0, \\ \omega_1^1 &= -\omega_2^2 = \omega_3^3 = -\omega_4^4 = \frac{1}{4}f'(u)\omega_2^4.\end{aligned}\quad (3.27)$$

Taking exterior derivatives of (3.27) by means of (1.4), (3.24) and (3.27), we arrive at the following system of exterior quadratic equations:

$$\begin{aligned}-\omega_1^3 \wedge \omega_3^2 - \omega_1^4 \wedge \omega_4^2 + \Omega_1^2 &= \frac{1}{4}(f'(u))^2\omega_1^4 \wedge \omega_2^4, \\ -\omega_2^3 \wedge \omega_3^1 - \omega_1^4 \wedge \omega_4^2 + \Omega_2^1 &= 0, \\ -\omega_3^1 \wedge \omega_1^4 - \omega_3^2 \wedge \omega_2^4 + \Omega_4^3 &= 0, \\ -\omega_4^1 \wedge \omega_1^3 - \omega_2^4 \wedge \omega_2^3 + \Omega_4^3 &= \frac{1}{2}f''(u)\omega_1^4 \wedge \omega_2^3 + \frac{1}{4}(f'(u))^2\omega_2^4 \wedge \omega_2^3, \\ -\omega_1^3 \wedge \omega_3^1 - \omega_1^4 \wedge \omega_4^1 + \Omega_1^1 &= \frac{1}{4}f''(u)\omega_1^4 \wedge \omega_2^4, \\ -\omega_3^1 \wedge \omega_1^3 - \omega_4^1 \wedge \omega_4^1 + \Omega_3^3 &= \frac{1}{4}f''(u)\omega_1^4 \wedge \omega_2^4,\end{aligned}\quad (3.28)$$

where Ω_α^β and Ω_i^j are the curvature 2-forms defined by (1.5). Substituting the values of these 2-forms from (1.5) and the values of the forms ω_i^α from (3.3) into equations (3.28) and equating coefficients in independent exterior forms $\omega_\alpha^i \wedge \omega_\beta^j$ to 0, in addition to equations (3.10)–(3.12) (which hold for any $AG(1, 3)$ -structure) we obtain the following additional conditions:

$$\begin{aligned}b_{243}^{111} &= b_{243}^{211} = b_{243}^{112} = b_{243}^{121} = b_{134}^{112} = b_{134}^{211} = b_{134}^{121} = b_{243}^{122} = 0, \\ b_{134}^{122} &= b_{134}^{212} = b_{134}^{221} = b_{134}^{222} = b_{333}^{412} = b_{433}^{412} = b_{334}^{412} = b_{343}^{412} = 0, \\ b_{433}^{321} &= b_{334}^{321} = b_{343}^{321} = b_{344}^{412} = b_{444}^{321} = 0, \\ b_{344}^{321} &= b_{434}^{321} = b_{443}^{321} = -\frac{1}{4}f''(u),\end{aligned}$$

that is, the only nonvanishing components of the object $b = \{b^i, b^2\}$ are the components b_{434}^{312} , b_{434}^{321} and $b_{344}^{312} = -b_{444}^{412}$.

In addition, we find that all coefficients of (3.3), except E_4 and C_4 , equal to 0, and for the coefficients E_4 and C_4 we find the following values:

$$C_4 = -\frac{1}{4}f''(u), \quad E_4 = -\frac{1}{4}(f'(u))^2.$$

As a result, equations (3.3) become

$$\begin{aligned}\omega_3^1 &= 0, & \omega_4^1 &= -\frac{1}{4}f''(u)\omega_2^4, \\ \omega_2^3 &= 0, & \omega_4^2 &= -\frac{1}{4}f''(u)\omega_1^4 - \frac{1}{4}(f'(u))^2\omega_2^4,\end{aligned}\quad (3.29)$$

and the curvature 2-forms (1.7) become

$$\begin{aligned}\Omega_1^1 &= \Omega_2^2 = \Omega_1^2 = \Omega_2^1 = \Omega_3^3 = 0, \\ \Omega_3^3 &= -\Omega_4^4 = \frac{1}{4}f''(u)\omega_1^4 \wedge \omega_2^4, \\ \Omega_4^3 &= \frac{1}{4}f''(u)\omega_1^3 \wedge \omega_2^4 - \frac{1}{4}f''(u)\omega_2^3 \wedge \omega_1^4.\end{aligned}\quad (3.30)$$

Since $b_{\alpha kl}^{\beta\gamma\delta} = 0$, by (2.41), we have $b_{\beta}^2 = 0$, and the almost Grassmann structure $AG(1, 3)$ under consideration is β -semiintegrable. This structure is not locally flat: in fact, $b_{(333)}^{312} = b_{(434)}^{312} = \frac{1}{4}f''(u) \neq 0$ for a general function $f(u)$, and thus $b_{\alpha}^1 \neq 0$, that is, the structure is not α -semiintegrable. Note that $f''(u) = 0$ if and only if $f(u) = au + b$, where a and b are constants. In this case the structure is β -semiintegrable, and consequently it is locally flat.

As we noted earlier, a structure $AG(1, 3)$ is equivalent to a conformal $CO(2, 2)$ -structure. Note that the $CO(2, 2)$ -structure corresponding to the β -semiintegrable structure $AG(1, 3)$ we have constructed in this example is self-dual.

By proving the local existence of a β -semiintegrable structure $AG(1, 3)$, we also proved a local existence of a self-dual $CO(2, 2)$ -structure. On the global existence of four-dimensional semiintegrable smooth compact oriented Riemannian manifolds see [11] and [10].

We will find now the metric of this $CO(2, 2)$ -structure.

To this end, we recall that the equation of the Segre cone of the $AG(1, 3)$ -structure (or the asymptotic cone of the corresponding $CO(2, 2)$ -structure) has the form (1.2) or $\omega_1^3\omega_2^4 - \omega_1^4\omega_2^3 = 0$. Thus the fundamental form of the $CO(2, 2)$ -structure is

$$g = \omega_1^3\omega_2^4 - \omega_1^4\omega_2^3,$$

or by (3.14),

$$g = dx dv + f(u) dy dv - dy du. \quad (3.31)$$

The quadratic form (3.31) defines on the manifold M^4 a pseudo-Riemannian metric of signature $(2, 2)$ that is conformally equivalent to the almost Grassmann structure $AG(1, 3)$ with the basis forms (3.24).

Let us apply the second method to prove that the structure (3.24) is β -semiintegrable and to find conditions under which this structure is α -semiintegrable.

For the structure (3.24), the equations (3.21) take the form

$$\mu dy + dx + f(u) dy = 0, \quad \mu dv + du = 0, \quad (3.32)$$

where $dy \wedge dv \neq 0$. Exterior differentiation of (3.32) gives

$$(d\mu + f'(u) du) \wedge dy = 0, \quad (d\mu + f'(u) du) \wedge dv = 0. \quad (3.33)$$

Since $dy \wedge dv \neq 0$, it follows from (3.33) that

$$d\mu + f'(u) du = 0. \quad (3.34)$$

The solution of (3.34) is

$$\mu = C_1 - f(u), \quad (3.35)$$

where C_1 is an arbitrary constant. Substituting this value of μ into equations (3.32), we find the following differential equations of submanifolds V_{β} :

$$C_1 dy + dx = 0, \quad (C_1 - f(u)) dv + du = 0. \quad (3.36)$$

By integration we find the following *closed form equations of submanifolds* V_β :

$$x = C_2 - C_1 y, \quad v = C_3 - \int \frac{du}{C_1 - f(u)}, \quad (3.37)$$

where C_2 and C_3 are arbitrary constants.

Thus we proved again the structure $AG(1, 3)$ with the basis forms (3.14) is β -semiintegrable. In addition, we proved that *the integral submanifolds V_β of this structure are defined by equations (3.37), the family of these submanifolds depends on three parameters C_1, C_2 and C_3 , and through any point (x_0, y_0, u_0, v_0) of M^4 there passes a one-parameter family of submanifolds V_β .*

To find conditions of α -semiintegrability of this structure, we first specialize equations (3.20) for it:

$$\lambda(dx + f(u) dy) + du = 0, \quad \lambda dy + dv = 0, \quad (3.38)$$

where $dx \wedge dy \neq 0$. Exterior differentiation of (3.38) gives

$$(d\lambda + \lambda^2 f'(u) dy) \wedge dx = 0, \quad (d\lambda + \lambda^2 f'(u) dy) \wedge dy = 0. \quad (3.39)$$

Since $dx \wedge dy \neq 0$, it follows from (3.39) that

$$d\lambda + \lambda^2 f'(u) dy = 0. \quad (3.40)$$

Exterior differentiation of (3.40) leads to the following equation:

$$\lambda^3 f''(u) dx \wedge dy = 0. \quad (3.41)$$

Since $dx \wedge dy \neq 0$, in general the structure (3.24) is not α -semiintegrable. It will be α -semiintegrable if and only if $f''(u) = 0$, or

$$f(u) = au + b. \quad (3.42)$$

Substituting $f(u) = au + b$ into equation (3.40), we find that

$$d\lambda + \lambda^2 a dy = 0. \quad (3.43)$$

The solution of (3.43) is

$$\lambda = \frac{1}{ay + C_4}, \quad (3.44)$$

where C_4 is an arbitrary constant. Substituting $f(u) = au + b$ and λ from (3.44) into equations (3.38), we find the following differential equations of submanifolds V_α :

$$dx + a d(uy) + b dy + C_4 du = 0, \quad dy + (ay + C_4) dv = 0. \quad (3.45)$$

By integration we find the following *closed form equations of submanifolds* V_α :

$$x + au y + b y + C_4 y = C_5, \quad \frac{1}{a} \log(ay + C_4) + v = C_6, \quad (3.46)$$

where C_5 and C_6 are arbitrary constants.

Thus, we proved again that in general the structure $AG(1, 3)$ with the basis forms (3.24) is not α -semiintegrable, and that it will be α -semiintegrable and consequently integrable if and only if $f(u) = au + b$. If it is the case, the closed form equations of submanifolds V_α and V_β are equations (3.46) and (3.37). If $f(u) = au + b$, the family of submanifolds V_α on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.24) depends on three parameters C_4 , C_5 and C_6 , and through any point (x_0, y_0, u_0, v_0) of M^4 there passes a one-parameter family of submanifolds V_α .

Let us indicate three more examples of almost Grassmann structures similar to the structure (3.24).

Example 3.6.

$$\begin{aligned}\omega_1^3 &= dx, & \omega_2^3 &= dy, \\ \omega_1^4 &= du + g(x) dv, & \omega_2^4 &= dv.\end{aligned}\tag{3.47}$$

Example 3.7.

$$\begin{aligned}\omega_1^3 &= dx, & \omega_2^3 &= h(v) dx + dy, \\ \omega_1^4 &= du, & \omega_2^4 &= dv.\end{aligned}\tag{3.48}$$

Example 3.8.

$$\begin{aligned}\omega_1^3 &= dx, & \omega_2^3 &= dy, \\ \omega_1^4 &= du, & \omega_2^4 &= k(y) du + dv.\end{aligned}\tag{3.49}$$

As was the case for the structure (3.24), each of the structures (3.47)–(3.49) is β -semiintegrable and in general not α -semiintegrable. These structures will be α -semiintegrable and consequently integrable if and only if the functions $g(x)$, $h(v)$ and $k(y)$ are linear functions of x , v , and y , respectively.

If we apply formulas (3.18) to Examples 3.5–3.8, then, we obtain that $C_\beta = 0$ for all these examples (that is, the corresponding structures are β -semiintegrable), and that the only nonvanishing components of the tensor C_α are a_3 for Examples 3.5 ($a_3 = -f''(u)/4$) and 3.7 ($a_3 = h''(v)/4$) and a_1 for Examples 3.6 ($a_1 = g''(x)/4$) and 3.8 ($a_1 = -k''(y)/4$). It follows that in the first case the polynomial C_α has the triple root $\lambda_1 = \lambda_2 = \lambda_3 = \infty$ and the simple root $\lambda_4 = 0$, and in the second case it has the simple root $\lambda_1 = \infty$ and the triple root $\lambda_2 = \lambda_3 = \lambda_4 = 0$. According to Section 5.4 of the book [1], this means that the fiber bundle E_α has two invariant distributions $\Delta_\alpha(\infty)$ and $\Delta_\alpha(0)$, and the distribution corresponding to a multiple root is integrable. Moreover, it is easy to prove that the distribution corresponding to a simple root is also integrable. For all cases the integral submanifolds V_α are defined by the equations $u = C_3$, $v = C_4$ (for $\lambda = \infty$) and by the equations $x = C_1$, $y = C_2$ (for $\lambda = 0$).

The next example was considered in [5]. However, since results obtained in [5] contain inaccuracies and the result is wrong (according to [5], the structure of this example is of general type), we give here a complete investigation of this example.

Example 3.9. Suppose again that x, y, u and v are coordinates in M^4 , and that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned}\omega_1^3 &= dx + f(u) dy, & \omega_2^3 &= dy, \\ \omega_1^4 &= du + g(y) dv, & \omega_2^4 &= dv.\end{aligned}\quad (3.50)$$

The metric of the $CO(2, 2)$ -structure which is equivalent to the $AG(1, 3)$ -structure with basis forms (3.40) is

$$g = dx dv - dy du + (f(u) - g(y)) dy dv. \quad (3.51)$$

In this case we have

$$\begin{cases} \omega_1^2 = g'(y) \omega_2^3 + \frac{1}{2} f'(u) \omega_1^4 - \frac{1}{2} f'(u) g(y) \omega_2^4, \\ \omega_4^3 = \frac{1}{2} f'(u) \omega_2^3, & \omega_2^1 = 0, & \omega_3^4 = 0, \\ \omega_1^1 = -\omega_2^2 = \omega_3^3 = -\omega_4^4 = \frac{1}{4} f'(u) \omega_2^4; \end{cases} \quad (3.52)$$

$$C_4 = -\frac{1}{4} f''(u), \quad E_4 = \frac{1}{2} f''(u) g(y) - \frac{1}{4} (f'(u))^2; \quad (3.53)$$

$$b_{434}^{312} = -b_{434}^{321} = \frac{1}{4} f''(u), \quad b_{344}^{312} = -b_{444}^{412} = \frac{1}{4} f''(u); \quad (3.54)$$

$$\begin{aligned}\omega_3^1 &= 0, & \omega_4^1 &= -\frac{1}{4} f''(u) \omega_2^4, \\ \omega_3^2 &= 0, & \omega_4^2 &= -\frac{1}{4} f''(u) \omega_1^4 + [\frac{1}{2} f''(u) g(y) - \frac{1}{4} (f'(u))^2] \omega_2^4;\end{aligned}\quad (3.55)$$

$$\begin{cases} \Omega_1^1 = \Omega_2^2 = \Omega_1^2 = \Omega_2^1 = \Omega_3^3 = 0, \\ \Omega_3^3 = -\Omega_4^4 = \frac{1}{4} f''(u) \omega_1^4 \wedge \omega_2^4, \\ \Omega_4^3 = \frac{1}{4} f''(u) \omega_1^3 \wedge \omega_2^4 - \frac{1}{4} f''(u) \omega_2^3 \wedge \omega_1^4. \end{cases} \quad (3.56)$$

We have $b_{\alpha kl}^{\beta \gamma \delta} = 0$, and by (2.41), $b_\alpha^1 = 0$, and $-b_{444}^{412} = b_{(344)}^{312} = -\frac{1}{4} f''(u) \neq 0$. This implies that $b_\beta^2 \neq 0$, and the almost Grassmann structure $AG(1, 3)$ with the basis forms (3.50) is β -semiintegrable but not locally flat.

If we apply the method of direct integration, we find that equations (3.21) take the form

$$\mu dy + dx + f(u) dy = 0, \quad \mu dv + du + g(y) dv = 0,$$

where $dy \wedge dv \neq 0$. Exterior differentiation of these equations leads to

$$(d\mu + f'(u) du) \wedge dy = 0, \quad (d\mu + g'(y) dy) \wedge dv = 0.$$

It follows that $d\mu + f'(u) du + g'(y) dy = 0$. By integration of the first equation, we find that

$$\mu = C_1 - f(u) - g(y), \quad (3.57)$$

where C_1 is a constant, and the following closed form equations of submanifolds V_β :

$$x = \int g(y) dy - C_1 y + C_2, \quad v = C_3 - \int \frac{du}{C_1 - f(u)}, \quad (3.58)$$

where C_2 and C_3 are constants. Hence we proved again that *the almost Grassmann structure with the basis forms (3.59) is β -semiintegrable*. In addition we proved that *the family of submanifolds V_α on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.50) depends on three parameters C_1 , C_2 and C_3 , and through any point (x_0, y_0, u_0, v_0) of M^4 there passes a one-parameter family of submanifolds V_α .*

If we look for conditions for α -semiintegrability of this structure, then after writing equations (3.20) and taking their exterior derivatives we come to equations (3.39) which imply (3.40) and (3.41). So this structure is α -semiintegrable and subsequently integrable if and only if $f(u) = au + b$. Moreover, the expression for λ and the closed form equations of submanifolds V_α are (3.44) and (3.46).

Note that if $g(y) = 0$, then equations (3.57) and (3.58) coincide with equations (3.35) and (3.37).

Note that the application of formulas (3.18) gives the same values for the 10 components of the tensor of conformal curvature that were obtained in Example 3.5.

Next we consider an example of an α -semiintegrable almost Grassmann structure $AG(1, 3)$.

Example 3.10. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned} \omega_1^3 &= dx + p(y) du, & \omega_2^3 &= dy, \\ \omega_1^4 &= du, & \omega_2^4 &= dv. \end{aligned} \quad (3.59)$$

The metric of the $CO(2, 2)$ -structure which is equivalent to the $AG(1, 3)$ -structure with basis forms (3.59) is

$$g = dx dv - dy du + p(y) du dv. \quad (3.60)$$

In this case we have

$$\begin{cases} \omega_1^2 = -\frac{1}{2}p'(y)\omega_1^4, & \omega_2^1 = 0, \\ \omega_4^3 = -\frac{1}{2}p'(y)\omega_2^3, & \omega_3^4 = 0, \\ \omega_1^1 = -\omega_2^2 = \omega_3^3 = -\omega_4^4 = -\frac{1}{4}p'(y)\omega_2^4; \end{cases} \quad (3.61)$$

$$B_4 = -\frac{1}{4}p''(y), \quad E_4 = -\frac{1}{4}(p'(y))^2; \quad (3.62)$$

$$b_{134}^{122} = b_{134}^{212} = b_{134}^{221} = -\frac{1}{4}p''(y); \quad (3.63)$$

$$\begin{aligned} \omega_3^1 &= 0, & \omega_4^1 &= 0, \\ \omega_3^2 &= -\frac{1}{4}p''(y)\omega_2^4, & \omega_4^2 &= -\frac{1}{4}p''(y)\omega_2^3 - \frac{1}{4}(p'(y))^2\omega_2^4; \end{aligned} \quad (3.64)$$

$$\begin{cases} \Omega_1^1 = \Omega_2^2 = \frac{1}{4}p''(y)\omega_2^3 \wedge \omega_2^4, \\ \Omega_2^1 = \Omega_3^4 = \Omega_4^3 = \Omega_3^3 = \Omega_4^4 = 0, \\ \Omega_1^2 = -\frac{1}{4}p''(y)(\omega_1^3 \wedge \omega_2^4 + \omega_2^3 \wedge \omega_1^4). \end{cases} \quad (3.65)$$

In this case $b_{jkl}^{i\gamma\delta} = 0$, that is, $b_{\alpha}^1 = 0$, and the almost Grassmann structure $AG(1, 3)$ with the basis forms (3.59) is α -semiintegrable. However, in this case $b_{\alpha kl}^{\beta\gamma\delta} \neq 0$ since we have $b_{134}^{212} = -\frac{1}{4}p''(y)$, that is $b_{\alpha}^1 \neq 0$, and in general the almost Grassmann structure $AG(1, 3)$ with the basis forms (3.59) is not β -semiintegrable, and consequently, it is not locally flat.

If we apply the second method to the structure (3.59), we find that

$$\lambda = -\frac{1}{p(y) + C_1}, \quad (3.66)$$

and the following closed form equations of submanifolds V_{α} :

$$x = C_2 - C_1 u, \quad v = C_3 + \int \frac{dy}{p(y) + C_3}, \quad (3.67)$$

where C_1, C_2 and C_3 are constants.

Thus the family of submanifolds V_{α} on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.59) depends on three parameters C_1, C_2 and C_3 , and through any point (x_0, y_0, u_0, v_0) of M^4 there passes an one-parameter family of submanifolds V_{α} .

If $p(y) = ay + b$, then we find that

$$\mu = au + C_4, \quad (3.68)$$

and the following closed form equations of submanifolds V_{β} :

$$x = C_5 - ayu - C_4 y - bu, \quad v = C_6 - \frac{1}{a}(au + C_4). \quad (3.69)$$

Thus if $p(y) = ay + b$, then the family of submanifolds V_{α} on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.49) depends on three parameters C_4, C_5 and C_6 , and through any point (x_0, y_0, u_0, v_0) of M^4 there passes a one-parameter family of submanifolds V_{α} .

Let us indicate examples of three more almost Grassmann structures similar to the structure (3.59).

Example 3.11.

$$\begin{aligned} \omega_1^3 &= dx, & \omega_2^3 &= dy + q(x) dv, \\ \omega_1^4 &= du, & \omega_2^4 &= dv. \end{aligned} \quad (3.70)$$

Example 3.12.

$$\begin{aligned} \omega_1^3 &= dx, & \omega_2^3 &= dy, \\ \omega_1^4 &= du + r(v) dx, & \omega_2^4 &= dv. \end{aligned} \quad (3.71)$$

Example 3.13.

$$\begin{aligned} \omega_1^3 &= dx, & \omega_2^3 &= dy, \\ \omega_1^4 &= du, & \omega_2^4 &= dv + s(u) dy. \end{aligned} \quad (3.72)$$

As was the case for the structure (3.59), each of the structures (3.70)–(3.72) is α -semiintegrable and in general not β -semiintegrable. These structures will be β -semiintegrable and

consequently integrable if and only if the functions $q(x)$, $r(v)$ and $s(u)$ are linear functions of x , v , and u , respectively.

If we apply formulas (3.18) to Examples 3.10–3.13, we obtain that $C_\alpha = 0$ for all these examples (that is, the corresponding structures are α -semiintegrable), and that the only non-vanishing components of the tensor C_β are b_3 for Examples 3.10 ($b_3 = -p''(y)/4$) and 3.12 ($b_3 = r''(v)/4$) and b_1 for Examples 3.11 ($b_1 = q''(x)/4$) and 3.13 ($b_1 = -s''(u)/4$). It follows that in the first case the polynomial C_β has the triple root $\mu_1 = \mu_2 = \mu_3 = \infty$ and the simple root $\mu_4 = 0$, and in the second case it has the simple root $\mu_1 = \infty$ and the triple root $\mu_2 = \mu_3 = \mu_4 = 0$. According to Section 5.4 of the book [1], this means that the fiber bundle E_β has two invariant distributions $\Delta_\beta(\infty)$ and $\Delta_\beta(0)$, and the distribution corresponding to a multiple root is integrable. Moreover, it is easy to prove that the distribution corresponding to a simple root is also integrable. For all cases the integral submanifolds V_β are defined by the equations $y = C_2$, $v = C_4$ (for $\mu = \infty$) and by the equations $x = C_1$, $u = C_3$ (for $\mu = 0$).

4. An almost Grassmann structure is associated with a web, and if a web is transversally geodesic or isoclinic, the corresponding almost Grassmann structure is α - or β -semiintegrable (see [1, Ch. 7]). Our next two examples are generated by examples of exceptional (nonalgebraizable) isoclinic webs of maximum 2-rank (see [7, 8] or [9, Ch. 8], and [1, §5.5]).

Example 3.14. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned} \omega_1^3 &= dx, & \omega_2^3 &= du, \\ \omega_1^4 &= -(y+v)dx + (u-x)dy, & \omega_2^4 &= (y+v)du + (u-x)dv. \end{aligned} \quad (3.73)$$

The metric of the $CO(2, 2)$ -structure which is equivalent to the $AG(1, 3)$ -structure with basis forms (3.73) is

$$g = 2(y+v)dxdu + (u-x)(dx dv - dy du). \quad (3.74)$$

In this case we have

$$\left\{ \begin{aligned} \omega_1^2 &= \frac{1}{2(u-x)} \omega_1^3, \\ \omega_2^1 &= -\frac{1}{2(u-x)} \omega_2^3, \\ \omega_1^1 &= -\omega_2^2 = \frac{3}{4(u-x)} (\omega_1^3 + \omega_2^3), \\ \omega_3^4 &= -\frac{2(y+v)}{u-x} (\omega_1^3 + \omega_2^3) + \frac{1}{2(u-x)} (-\omega_1^4 + \omega_2^4), \\ \omega_4^3 &= 0, \\ \omega_3^3 &= -\omega_4^4 = \frac{1}{4(u-x)} (-\omega_1^3 + \omega_2^3); \end{aligned} \right. \quad (3.75)$$

$$A_1 = B_2 = -\frac{5}{4(u-x)^2}; \quad A_2 = -\frac{7}{4(u-x)^2}; \quad (3.76)$$

$$b_{333}^{412} = -\frac{8(y+v)}{(u-x)^2}; \quad (3.77)$$

$$\begin{aligned} \omega_3^1 &= -\frac{1}{4(u-x)^2} (5\omega_1^3 + 7\omega_2^3), & \omega_4^1 &= 0, \\ \omega_3^2 &= -\frac{1}{4(u-x)^2} (7\omega_1^3 + 5\omega_2^3), & \omega_4^2 &= 0; \end{aligned} \quad (3.78)$$

$$\begin{aligned} \Omega_1^2 &= \Omega_2^1 = \Omega_1^1 = \Omega_2^2 = \Omega_4^3 = \Omega_3^3 = \Omega_4^4 = 0, \\ \Omega_3^4 &= -\frac{8(y+v)}{(u-x)^2} \omega_1^3 \wedge \omega_2^3. \end{aligned} \quad (3.79)$$

It is easy to check that $b_{\alpha kl}^{(\beta\gamma\delta)} = 0$, that is, $b_\beta^2 = 0$, and the almost Grassmann structure $AG(1, 3)$ with basis forms (3.73) is β -semiintegrable. However, in this case $b_{(jkl)}^{i\gamma\delta} \neq 0$. Thus, the almost Grassmann structure $AG(1, 3)$ with basis forms (3.73) is not locally flat.

If we apply the second method for the structure (3.73), then while looking for α -semiintegrability conditions, we find that

$$d\lambda = dy + dv - \frac{y+v+\lambda}{u-x} dx + \frac{y+v-\lambda}{u-x} du.$$

Exterior differentiation of this equation gives $\lambda dx \wedge du = 0$. Since on V_α we have $dx \wedge du \neq 0$, the structure (3.73) is never α -semiintegrable.

For β -semiintegrability we find that

$$\begin{aligned} d\mu &= \frac{2\mu(1-\mu) du}{u-x}, & dx + \mu du &= 0, \\ dy &= -\mu dv - \frac{2\mu(y+v)}{u-x} du. \end{aligned} \quad (3.80)$$

It is easy to see from (3.80) that

$$\frac{d\mu}{2\mu(1-\mu)} = \frac{du}{u-x} = \frac{dx}{-\mu(u-x)},$$

and subsequently

$$\frac{d(u-x)}{(u-x)(1+\mu)} = \frac{d\mu}{2\mu(1-\mu)}. \quad (3.81)$$

The solution of equation (3.81) is

$$\frac{\sqrt{\mu}}{\mu-1} = C_1(u-x). \quad (3.82)$$

The submanifolds V_β of the distribution Δ_β are defined by the completely integrable system (3.80) where μ can be found from equations (3.82).

Thus the family of submanifolds V_β on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.63) depends on three parameters, and through any point (x_0, y_0, u_0, v_0) of M^4 there passes an one-parameter family of submanifolds V_α .

If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent $CO(2, 2)$ -structure we have $C_\beta = 0$ and the only nonvanishing component of C_α is $a_0 = -(8(y+v)/(u-x)^2)$. This means that the polynomial $C_\alpha(\lambda)$ has a quadruple root $\lambda = 0$, and the fiber bundle E_α possesses an integrable distribution $\Delta_\alpha(0)$. It is easy to prove that the submanifolds V_α of this distribution are defined by the equations $x = C_1, u = C_3$, where C_1 and C_3 are constants.

Example 3.15. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned}\omega_1^3 &= dx, & \omega_2^3 &= du, \\ \omega_1^4 &= -v dx + u dy, & \omega_2^4 &= y du - x dv.\end{aligned}\quad (3.83)$$

The metric of the $CO(2, 2)$ -structure which is equivalent to the $AG(1, 3)$ -structure with basis forms (3.83) is

$$g = (y + v) dx du - x dx dv - u dy du. \quad (3.84)$$

In this case we have

$$\left\{ \begin{aligned} \omega_1^2 &= -\frac{1}{2x} \omega_1^3, & \omega_3^4 &= \left(\frac{y}{x} - \frac{v}{u}\right) (\omega_1^3 + \omega_2^3) - \frac{1}{2u} \omega_1^4 - \frac{1}{2x} \omega_2^4, \\ \omega_2^1 &= -\frac{1}{2u} \omega_2^3, & \omega_1^1 &= -\omega_2^2 = \left(\frac{1}{4u} - \frac{1}{2x}\right) \omega_1^3 + \left(\frac{1}{2u} - \frac{1}{4x}\right) \omega_2^3, \\ \omega_4^3 &= 0, & \omega_3^3 &= -\omega_4^4 = \left(\frac{1}{4u} + \frac{1}{2x}\right) \omega_1^3 + \left(\frac{1}{2u} + \frac{1}{4x}\right) \omega_2^3; \end{aligned} \right. \quad (3.85)$$

$$A_1 = \frac{1}{4u^2} + \frac{1}{4xu}, \quad A_2 = \frac{1}{4xu} - \frac{1}{4x^2} - \frac{1}{4u^2}, \quad B_2 = \frac{1}{2xu} - \frac{1}{4x^2}; \quad (3.86)$$

$$b_{333}^{412} = \left(\frac{1}{u} - \frac{1}{x}\right) \left(\frac{y}{x} - \frac{v}{u}\right), \quad b_{333}^{312} = b_{334}^{412} = -b_{343}^{412} = \frac{1}{4x^2} - \frac{1}{4u^2}; \quad (3.87)$$

$$\begin{aligned}\omega_3^1 &= \left(-\frac{1}{4u^2} + \frac{1}{2xu}\right) \omega_1^3 + \left(\frac{1}{4xu} - \frac{1}{4x^2} - \frac{1}{4u^2}\right) \omega_2^3, & \omega_4^1 &= 0, \\ \omega_3^2 &= \left(\frac{1}{4xu} - \frac{1}{4x^2} - \frac{1}{4u^2}\right) \omega_1^3 + \left(\frac{1}{2xu} - \frac{1}{4x^2}\right) \omega_2^3, & \omega_4^2 &= 0;\end{aligned}\quad (3.88)$$

$$\left\{ \begin{aligned} \Omega_1^1 &= \Omega_2^2 = \Omega_1^2 = \Omega_2^1 = \Omega_4^3 = 0, \\ \Omega_3^3 &= -\Omega_4^4 = \left(\frac{1}{4u^2} - \frac{1}{4x^2}\right) \omega_1^3 \wedge \omega_2^3, \\ \Omega_3^4 &= \left(\frac{1}{u} - \frac{1}{x}\right) \left(\frac{y}{x} - \frac{v}{u}\right) \omega_1^3 \wedge \omega_2^3 + \left(\frac{1}{4x^2} - \frac{1}{4u^2}\right) (\omega_1^3 \wedge \omega_2^4 - \omega_2^3 \wedge \omega_1^4). \end{aligned} \right. \quad (3.89)$$

It is easy to check that $b_{\alpha kl}^{(\beta\gamma\delta)} = 0$, that is, $b_\beta^2 = 0$, and *the almost Grassmann structure* $AG(1, 3)$ *with the basis forms* (3.83) *is β -semiintegrable*. However, in this case $b_{(jkl)}^{i\gamma\delta} \neq 0$. Thus, *the almost Grassmann structure* $AG(1, 3)$ *with the basis forms* (3.83) *is not locally flat*.

If we apply the second method for the structure (3.73), then while looking for α -semiintegrability conditions, we find that

$$d\lambda = \frac{y + \lambda}{u} dx - dy + \frac{v - \lambda}{u} du + dv. \quad (3.90)$$

Exterior differentiation of (3.90) gives

$$\lambda = -\frac{1}{x^2 + u^2} [vx(u - x) - yu(x + u)]. \quad (3.91)$$

The differential equations of submanifolds V_α have the form

$$\begin{aligned} (v + y)(x + u) dx - (x^2 + u^2) dy &= 0, \\ (v + y)(x + u) du - (x^2 + u^2) dv &= 0. \end{aligned} \quad (3.92)$$

It is easy to prove that the integrability conditions of equations (3.92) implies that du is proportional to dx . This is impossible since on V_α we have $dx \wedge du \neq 0$. Thus, *the structure* $AG(1, 3)$ *with the basis forms* (3.83) *is never α -semiintegrable*.

The β -submanifolds of this structure are defined by the completely integrable system

$$\begin{aligned} \mu du + dx &= 0, \\ \mu[(y + v) du - x dv] + u dy &= 0, \\ d \log(\mu - 1) &= \left(\frac{1}{u} - \frac{1}{x} \right) dx. \end{aligned} \quad (3.93)$$

Thus *the family of submanifolds* V_β *on the manifold* M^4 *carrying the* $AG(1, 3)$ -*structure with the basis forms* (3.83) *depends on three parameters, and through any point* (x_0, y_0, u_0, v_0) *of* M^4 *there passes an one-parameter family of submanifolds* V_β .

If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent $CO(2, 2)$ -structure we have $C_\beta = 0$ and the only nonvanishing components of C_α are

$$a_0 = \left(\frac{1}{u} - \frac{1}{x} \right) \left(\frac{y}{x} - \frac{v}{u} \right) \quad \text{and} \quad a_1 = \frac{1}{4} \left(\frac{1}{x^2} - \frac{1}{u^2} \right).$$

This means that the polynomial $C_\alpha(\lambda)$ has a triple root $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and a simple root $\lambda_4 = (x + u)/(xv - yu)$, and the fiber bundle E_α possesses an integrable distribution $\Delta_\alpha(0)$. It is easy to prove that the integral submanifolds V_α of this distribution are defined by the equations $x = C_1, u = C_3$, where C_1 and C_3 are constants. It is also easy to check that the distribution $\Delta(\lambda_4)$ is not integrable.

The next example is generated by an example of an isoclinic three-web given in [3] (see Exercise 6 on p. 133).

Example 3.16. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 3)$ are

$$\begin{aligned}\omega_1^3 &= (y - v) dx + (x + u) dy, & \omega_2^3 &= (y - v) du - (x + u) dv, \\ \omega_1^4 &= dy, & \omega_2^4 &= dv.\end{aligned}\quad (3.94)$$

The metric of the $CO(2, 2)$ -structure which is equivalent to the $AG(1, 3)$ -structure with basis forms (3.94) is

$$g = (y - v) (dx dv - dy du) + 2(x + u) dy dv. \quad (3.95)$$

In this case we have

$$\left\{ \begin{aligned} \omega_1^2 &= -\frac{1}{2(y - v)} \omega_1^4, \\ \omega_2^1 &= \frac{1}{2(y - v)} \omega_2^3, \\ \omega_1^1 &= -\omega_2^2 = -\frac{3}{4(y - v)} (\omega_1^4 + \omega_2^4), \\ \omega_3^4 &= 0, \\ \omega_4^3 &= \frac{1}{2(y - v)} (\omega_1^3 - \omega_2^3) - \frac{2(x + u)}{y - v} (\omega_1^4 + \omega_2^4), \\ \omega_3^3 &= -\omega_4^4 = -\frac{1}{4(y - v)} (\omega_1^4 - \omega_2^4); \end{aligned} \right. \quad (3.96)$$

$$C_3 = -\frac{5}{4(y - v)^2}, \quad C_4 = -\frac{7}{4(y - v)^2}, \quad E_4 = -\frac{5}{4(y - v)^2}; \quad (3.97)$$

$$b_{444}^{312} = \frac{8(x + u)}{(y - v)^2}; \quad (3.98)$$

$$\omega_3^1 = 0, \quad \omega_4^1 = -\frac{1}{4(y - v)^2} (5\omega_1^4 + 7\omega_2^4), \quad (3.99)$$

$$\omega_3^2 = 0, \quad \omega_4^2 = -\frac{1}{4(y - v)^2} (7\omega_1^4 + 5\omega_2^4);$$

$$\begin{aligned}\Omega_1^1 &= \Omega_2^2 = \Omega_1^2 = \Omega_2^1 = \Omega_3^3 = \Omega_4^4 = \Omega_3^4 = \Omega_4^3 = 0, \\ \Omega_4^3 &= \frac{8(x + u)}{(y - v)^2} \omega_1^4 \wedge \omega_2^4.\end{aligned}\quad (3.100)$$

It is easy to check that $b_{\alpha kl}^{(\beta\gamma\delta)} = 0$, that is, $b_\beta = 0$, and the almost Grassmann structure $AG(1, 3)$ is β -semiintegrable. However, in this case $b_{(jkl)}^{i\gamma\delta} \neq 0$. Thus, the almost Grassmann structure is not locally flat.

If we apply the method of direct integration, we find that the submanifolds V_β are defined by the following completely integrable system of differential equations:

$$\begin{aligned}\frac{d\mu}{2\mu(\mu-1)} &= \frac{dv}{y-v}, \\ (y-v)(\mu du + dx) - 2\mu(x+u)dv &= 0, \\ \mu dv + dy &= 0.\end{aligned}\tag{3.101}$$

It is easy to see from (3.101) that

$$\frac{d\mu}{2\mu(\mu-1)} = \frac{dv}{y-v} = \frac{dy}{-\mu(y-v)},$$

and subsequently

$$\frac{d(y-v)}{(y-v)(1+\mu)} = \frac{d\mu}{2\mu(\mu-1)}.\tag{3.102}$$

The solution of equation (3.102) is

$$\frac{\mu-1}{\sqrt{\mu}} = C_1(y-v).\tag{3.103}$$

The submanifolds V_β of the distribution Δ_β are defined by the completely integrable system (3.101) where μ can be found from equations (3.103).

Thus the family of submanifolds V_β on the manifold M^4 carrying the $AG(1, 3)$ -structure with the basis forms (3.94) depends on three parameters, and through any point (x_0, y_0, u_0, v_0) of M^4 there passes an one-parameter family of submanifolds V_β .

If we apply formulas (3.18), we find that for the tensor of conformal curvature of the equivalent $CO(2, 2)$ -structure we have $C_\beta = 0$ and the only nonvanishing component of C_α is $a_4 = -8(x+u)/(y-v)^2$. This means that the polynomial $C_\alpha(\lambda)$ has a quadruple root $\lambda = \infty$, and the fiber bundle E_α possesses an integrable distribution $\Delta_\alpha(\infty)$. It is easy to prove that the submanifolds V_α of this distribution are defined by the equations $y = C_2, v = C_4$, where C_2 and C_4 are constants.

5. The next example is generated by an example of an almost Grassmann structure $AG(1, 4)$ associated with a six-dimensional Bol web considered in [3, p. 270].

Example 3.17. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, 4)$ are

$$\begin{aligned}\omega_1^3 &= dx, \\ \omega_1^4 &= [-2v - 2uz + (2x-1)w]e^{-2x}dx + dy + ue^{-2x}dz, \\ \omega_1^5 &= dz, \\ \omega_2^3 &= du, \\ \omega_2^4 &= (zdu + dv - xdw)e^{-2x}, \\ \omega_2^5 &= dw.\end{aligned}\tag{3.104}$$

We prove that the $AG(1, 4)$ -structure (3.114) is α -semiintegrable. We apply the method of the direct integration.

For the structure (3.104) equations (3.22) of surfaces V_α , $\dim V_\alpha = 2$ take the form

$$\begin{aligned} \{\lambda_1 + [-2v - 2uz + (2x - 1)w]e^{-2x}\} dx + dy + ue^{-2x} dz &= 0, \\ \lambda_1 du + (z du + dv - x dw)e^{-2x} &= 0, \\ \lambda_2 dx + dz &= 0, \\ \lambda_2 du + dw &= 0. \end{aligned} \quad (3.105)$$

Note that on V_α we have $dx \wedge du \neq 0$. Exterior differentiation of the last two equations of (3.105) gives the following exterior quadratic equations:

$$d\lambda_2 \wedge dx = 0, \quad d\lambda_2 \wedge du = 0.$$

Since $dx \wedge du \neq 0$, it follows that $d\lambda_2 = 0$ and

$$\lambda_2 = C_2, \quad (3.106)$$

where C_2 is a constant. Now the last two equations of (3.105) can be written as

$$dz = -C_2 dx = 0, \quad dw = -C_2 du. \quad (3.107)$$

Integration of (3.107) gives

$$z = -C_2 x + C_3, \quad w = -C_2 u + C_4, \quad (3.108)$$

where C_3 and C_4 are constants.

By (3.106) and (3.107), the first two equations of (3.105) become

$$\begin{aligned} dy &= \{-\lambda_1 + [2v + 2uz + (1 - 2x)w + C_2 u]e^{-2x}\} dx, \\ dv &= -(\lambda_1 e^{2x} + z + C_2 x) du. \end{aligned} \quad (3.109)$$

Exterior differentiation of (3.109) leads to the following exterior quadratic equations:

$$(d\lambda_1 + 2\lambda_1 du) \wedge dx = 0, \quad (d\lambda_1 + 2\lambda_1 dx) \wedge du = 0.$$

These equations can be written as

$$[d\lambda_1 + 2\lambda_1(dx + du)] \wedge dx = 0, \quad [d\lambda_1 + 2\lambda_1(dx + du)] \wedge du = 0.$$

Since $dx \wedge du \neq 0$, it follows that $d\lambda_1 + 2\lambda_1(dx + du) = 0$. This implies that

$$\lambda_1 = C_1 e^{-2(x+u)}. \quad (3.110)$$

Substituting this value of λ_1 and z from (3.108) into the second equation of (3.109), we find that $dv = -(C_1 e^{-2u} + C_3) du$. This implies that

$$v = \frac{1}{2} C_1 e^{-2u} - C_3 u + C_5, \quad (3.111)$$

where C_5 is a constant.

Substituting the values of λ_1 , z and w from (3.111), (3.110) and (3.108) into the first equation of (3.109), we find that $dy = [2C_5 + C_4(1 - 2x)]e^{-2x}dx$. The solution of this equation is

$$y = (-C_5 + C_4x)e^{-2x} + C_6, \quad (3.112)$$

where C_6 is a constant.

Thus two-dimensional surfaces V_α are defined by the closed form equations (3.108), (3.110), (3.111) and (3.112). This completes the proof that the $AG(1, 4)$ -structure (3.104) is α -semiintegrable. Thus the family of submanifolds V_α on the manifold M^6 carrying the $AG(1, 4)$ -structure with the basis forms (3.104) depends on six parameters, and through any point $(x_0, y_0, z_0, u_0, v_0, w_0)$ of M^6 there passes a two-parameter family of submanifolds V_α .

We will prove now that this structure is not β -semiintegrable. Equations (3.23) of surfaces V_β , $\dim V_\beta = 3$, can be written as follows:

$$\mu du + dx = 0, \quad \mu\omega_2^4 + \omega_1^4 = 0, \quad \mu dw + dz = 0. \quad (3.113)$$

where $\mu = -s_1/s_2$. Taking exterior derivatives of the first and third equations of (3.113), we find that $d\mu \wedge du = 0$ and $d\mu \wedge dw = 0$. Since on surfaces V_β we must have $du \wedge dv \wedge dw \neq 0$, it follows that $d\mu = 0$, and

$$\mu = C, \quad (3.114)$$

where C is a constant. Substituting this value of μ into the first and third equations of (3.113), we find that

$$dx = -C du, \quad dz = -C dw, \quad (3.115)$$

and

$$x = -Cu + C_1, \quad z = -Cw + C_2, \quad (3.116)$$

Substituting μ , x and z from (3.114) and (3.115) into the second equation of (3.113), we obtain

$$\begin{aligned} \frac{1}{C} e^{2(C_1 - Cu)} = -dv + [C_1 + u(1 - C)]dw \\ + [(C + 2C_1 - 1)w - 2v - 2C_2u - C_2]du. \end{aligned} \quad (3.117)$$

Taking exterior derivative of (3.117), we find that if the surfaces V_β exist, we would have

$$du \wedge [(2C - 1)dv + 2(1 - C)(Cu + C_1 - 2)]dw = 0. \quad (3.118)$$

This is impossible since (3.118) implies that du is a linear combination of dv and dw and as a result, $du \wedge dv \wedge dw = 0$.

Thus the $AG(1, 4)$ -structure is not β -semiintegrable.

Example 3.5 can be generalized in the following manner.

Example 3.18. Suppose that $x_\alpha, y_\alpha, \alpha = 1, \dots, p$, are coordinates in M^{2p} , and that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(p - 1, p + 1)$ are

$$\begin{aligned} \omega_1^{p+1} = dx_1 + f(y_1)dx_2, \quad \omega_2^{p+1} = dx_2, \quad \omega_s^{p+1} = dx_s, \\ \omega_1^{p+2} = dy_1, \quad \omega_2^{p+2} = dy_2, \quad \omega_s^{p+2} = dy_s, \end{aligned} \quad (3.119)$$

where $s = 3, \dots, p$. If the structure (3.119) is α -semiintegrable, then $dy_1 \wedge dy_2 \wedge \dots \wedge dy_p \neq 0$, and the rows of the matrix (ω_α^i) are proportional:

$$dx_1 + f(y) dx_2 + \lambda dy_1 = 0, \quad dx_2 + \lambda dy_2 = 0, \quad dx_s + \lambda dy_s = 0. \quad (3.120)$$

Exterior differentiation of (3.120) gives the following exterior quadratic equations:

$$f'(y_1) dy_1 \wedge dx_2 + d\lambda \wedge dy_1 = 0, \quad d\lambda \wedge dy_2 = 0, \quad d\lambda \wedge dy_s = 0. \quad (3.121)$$

The last two equations of (3.121) imply that $d\lambda = 0$. By (3.120), the first equation of (3.121) gives

$$\lambda f'(y_1) dy_1 \wedge dy_2 = 0.$$

Since $dy_1 \wedge dy_2 \neq 0$, it follows that $f'(y_1) = 0$ and $f(y_1) = a$, where a is a constant. Thus the structure (3.119) is α -semiintegrable if and only if $f(y_1)$ is constant. If it is the case, closed form equations of integral submanifolds V_α of this structure have the form

$$x_1 + ax_2 + by_1 = C_1, \quad x_2 + by_2 = C_2, \quad x_s + by_s = C_s. \quad (3.122)$$

Hence the submanifolds V_α are flat p -dimensional submanifolds, and the family of submanifolds V_α depends on $p + 1$ constants b, C_1, C_2 and C_s .

If the structure (3.129) is β -semiintegrable, then $dx_2 \wedge dy_2 \neq 0$, and the columns of the matrix (ω_α^i) are proportional:

$$\begin{aligned} dx_1 + f(y_1) dx_2 + \mu_1 dy_x &= 0, & dx_s + \mu_s dx_2 &= 0, \\ dy_1 + \mu_1 dy_2 &= 0, & dy_s + \mu_s dy_2 &= 0. \end{aligned} \quad (3.123)$$

Exterior differentiation of (3.123) give the following exterior quadratic equations:

$$\begin{aligned} (d\mu_1 + f'(y_1)) dy_1 \wedge dx_2 &= 0, & d\mu_s \wedge dx_2 &= 0, \\ (d\mu_1 + f'(y_1)) dy_1 \wedge dy_2 &= 0, & d\mu_s \wedge dy_2 &= 0. \end{aligned} \quad (3.124)$$

Since $dx_2 \wedge dy_2 \neq 0$, it follows from (3.114) that

$$d\mu_1 + f'(y_1) dy_1 = 0, \quad d\mu_s = 0,$$

and

$$\mu_1 + f(y_1) = C_1, \quad \mu_s = C_s. \quad (3.125)$$

Substituting (3.125) into system (3.123), we find that

$$\begin{aligned} dx_1 + C_1 dx_2 &= 0, & dx_s + C_s dx_2 &= 0, \\ dy_1 + (C_1 - f(y_1)) dy_2 &= 0, & dy_s + C_s dy_2 &= 0. \end{aligned} \quad (3.126)$$

The solution of this system is

$$\begin{aligned} x_1 + C_1 x_2 &= A_1, & x_s + C_s x_2 &= A_s, \\ \int \frac{dy_1}{C_1 - f(y_1)} + y_2 &= B_1, & y_s + C_s y_2 &= B_s. \end{aligned} \quad (3.127)$$

Hence the two-dimensional integral submanifolds V_β are defined by closed form equations (3.127), and the family of submanifolds V_β depends on $3(p-1)$ constants C_1, A_1, B_1, C_s, A_s and B_s .

Thus if $f(y_1)$ is not a constant, the structure $AG(p-1, p+1)$ with the structure forms (3.119) is β -semiintegrable, and if $f(y_1)$ is a constant, this structure is locally flat.

Example 3.10 can be also generalized.

Example 3.19. Suppose that $x^3, x^4, x^t, y^3, y^4, y^t, t = 5, \dots, q+2$, are coordinates in M^{2q} , and that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, q+1)$ are

$$\begin{aligned} \omega_1^3 &= dx^3 + p(y^3) dx^4, & \omega_2^{p+1} &= dy^3, \\ \omega_1^4 &= dx^4, & \omega_2^4 &= dy^4, \\ \omega_1^t &= dx^t, & \omega_2^t &= dy^t, \end{aligned} \quad (3.128)$$

where $t = 5, \dots, q+2$.

If the structure (3.128) is α -semiintegrable, then $dx^4 \wedge dy^4 \neq 0$ and the rows of the matrix (ω_α^i) are proportional:

$$\begin{aligned} dx^3 + p(y^3) dx^4 + \lambda^3 dx^4 &= 0, & dy^3 + \lambda^3 dy^4 &= 0, \\ dx^t + \lambda^t dx^4 &= 0, & dy^t + \lambda^t dy^4 &= 0. \end{aligned} \quad (3.129)$$

Exterior differentiation of (3.129) give the following exterior quadratic equations:

$$\begin{aligned} (d\lambda^3 + p'(y^3) dy^3) \wedge dx^4 &= 0, & (d\lambda^3 + p'(y^3) dy^3) \wedge dy^4 &= 0, \\ d\lambda^t \wedge dx^4 &= 0, & d\lambda^t \wedge dy^4 &= 0. \end{aligned} \quad (3.130)$$

Since $dy^3 \wedge dy^4 \neq 0$, equations (3.130) imply that $d\lambda^3 + p'(y^3) dy^3 = 0$ and $d\lambda^t = 0$. Thus we have

$$\lambda^3 + p(y^3) = C^3, \quad \lambda^t = C^t. \quad (3.131)$$

As a result, equations (3.129) become

$$\begin{aligned} dx^3 + C^3 dx^4 &= 0, & dy^3 + (C^3 - p(y^3)) dy^4 &= 0, \\ dx^t + C^t dx^4 &= 0, & dy^t + C^t dy^4 &= 0. \end{aligned} \quad (3.132)$$

It follows from (3.132) that closed form equations of two-dimensional integral submanifolds V_α are

$$\begin{aligned} x^3 + C^3 x^4 &= A^3, & \int \frac{dy^3}{C^3 - p(y^3)} + y^4 &= B^4, \\ x^t + C^t x^4 &= A^t, & y^t + C^t y^4 &= B^t. \end{aligned} \quad (3.133)$$

Thus the family of submanifolds V_α depends on $3(q-1)$ constants C^3, A^3, B^3, C^t, A^t and B^t .

If the structure (3.128) is β -semiintegrable, then $dy^3 \wedge dy^4 \wedge \dots \wedge dy^{q+2} \neq 0$, and the columns of the matrix (ω_α^i) are proportional:

$$dx^3 + p(y^3) dx^4 + \mu dy^3 = 0, \quad dx^4 + \mu dy^4 = 0, \quad dx^t + \mu dy^t = 0. \quad (3.134)$$

Exterior differentiation of (3.134) gives the following exterior quadratic equations:

$$p'(y^3) dy^3 \wedge dx^4 + d\mu \wedge dy^3 = 0, \quad d\mu \wedge dy^4 = 0, \quad d\mu \wedge dy^t = 0. \quad (3.135)$$

It follows from (3.135) that $d\mu = 0$ and consequently $\mu = C_0$. As a result, by (3.134), the first equation of (3.135) becomes

$$C_0 p'(y^3) dy^3 \wedge dy^4 = 0.$$

Since $dy^3 \wedge dy^4 \neq 0$, it follows that $p'(y^3) = 0$ and $p(y^3) = a$. Thus the structure (3.118) is β -semiintegrable if and only if $p(y^3)$ is a constant. If it is the case, equations (3.134) become

$$dx^3 + a dx^4 + C_0 dy^3 = 0, \quad dx^4 + C_0 dy^4 = 0, \quad dx^t + C_0 dy^t = 0, \quad (3.136)$$

and closed form equations of integral submanifolds V_β of this structure have the form

$$x^3 + ax^4 + C_0 y^3 = C^3, \quad x^4 + C_0 y^4 = C^4, \quad x^t + C_0 y^t = C^t. \quad (3.137)$$

Hence the submanifolds V_β are flat two-dimensional submanifolds, and the family of submanifolds V_β depends on $q + 2$ constants a, C^0, C^3, C^4 and C^t .

Thus if $p(y^3)$ is not a constant, the structure $AG(1, q + 1)$ with the structure forms (3.128) is α -semiintegrable, and if $p(y^3)$ is a constant, this structure is locally flat.

Example 3.17 can be generalized to an example of an α -integrable almost Grassmann structure $AG(1, q + 1)$.

Example 3.20. Suppose that the basis 1-forms ω_α^i of an almost Grassmann structure $AG(1, q + 1)$ are

$$\begin{aligned} \omega_1^3 &= dx, \\ \omega_1^4 &= [-2v - 2uz + (2x - 1)w] e^{-2x} dx + dy + ue^{-2x} dz, \\ \omega_1^s &= dz^s, \\ \omega_2^3 &= du, \\ \omega_2^4 &= (z du + dv - x dw) e^{-2x}, \\ \omega_2^s &= dw^s, \end{aligned} \quad (3.138)$$

where $s = 5, \dots, q + 2$.

Equations (3.105) will take the form

$$\begin{aligned} \{\lambda_1 + [-2v - 2uz + (2x - 1)w] e^{-2x}\} dx + dy + ue^{-2x} dz &= 0, \\ \lambda_1 du + (z du + dv - x dw) e^{-2x} &= 0, \\ \lambda_{s-3} dx + dz^s &= 0, \\ \lambda_{s-3} du + dw^s &= 0, \quad s = 5, \dots, q + 2. \end{aligned} \quad (3.139)$$

The proof of the fact that *the* $AG(1, q + 1)$ -structure with the structure forms (3.138) is α -semiintegrable is similar to that for Example 3.17.

Note that using this method of generalization, we cannot find a semiintegrable $AG(2, 5)$ -structure (in this case $p = q = 3$). In fact, if we set

$$\begin{aligned}\omega_1^4 &= \omega, & \omega_2^4 &= dx_2, & \omega_3^4 &= dx_3, \\ \omega_1^5 &= dy_1, & \omega_2^5 &= dy_2, & \omega_3^5 &= dy_3, \\ \omega_1^6 &= dz_1, & \omega_2^6 &= dz_2, & \omega_3^6 &= dz_3,\end{aligned}$$

then for α -semiintegrability we must have

$$\begin{aligned}\omega + \lambda dy_1 &= 0, & dx_2 + \lambda dy_2 &= 0, & dx_3 + \lambda dy_3 &= 0, \\ dz_1 + \tilde{\lambda} dy_1 &= 0, & dz_2 + \tilde{\lambda} dy_2 &= 0, & dz_3 + \tilde{\lambda} dy_3 &= 0.\end{aligned}$$

It is easy to find from this that $\lambda = C$, $\tilde{\lambda} = \tilde{C}$, where C and \tilde{C} are constants, $d\omega = 0$, that is, ω is a total differential, $\omega = dx_1$. But in this case the structure in question is also β -semiintegrable, i.e., this structure is locally flat.

References

- [1] M.A. Akivis and V.V. Goldberg, *Conformal Differential Geometry and Its Generalizations* (Wiley-Interscience, New York, 1996).
- [2] M.A. Akivis and V.V. Goldberg, On the theory of almost Grassmann structures, in: J. Szenthe, ed., *New Developments in Differential Geometry*, Proc. Conf. Budapest, Hungary, July 27–30, 1996 (Kluwer, Dordrecht, 1999) 1–37.
- [3] M.A. Akivis and A.M. Shelekhov, *Geometry and Algebra of Multidimensional Three-Webs* (Kluwer, Dordrecht, 1992).
- [4] É. Cartan, Les espaces à connexion conforme, *Ann. Soc. Polon. Math.* **2** (1923) 171–221; also É. Cartan, *Œuvres Complètes III. Divers, Géométrie, Différentielle*, V. 1–2 (Gauthier-Villars, Paris, 1955) 747–797.
- [5] P.F. Dhooghe, Grassmannianlike manifolds, in: *Geometry and Topology of Submanifolds V*, Leuven/Brussels, 1992 (World Sci. Publishing, River Edge, 1993) 147–160.
- [6] P.F. Dhooghe, Grassmannian structures on manifolds, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994) (1) 597–621.
- [7] V.V. Goldberg, 4-tissus isoclines exceptionnels de codimension deux et de 2-rang maximal, *C.R. Acad. Sci. Paris Sér. I Math.* **301** (1985) (11) 593–596.
- [8] V.V. Goldberg, Isoclinic webs $W(4, 2, 2)$ of maximum 2-rank, in: *Differential Geometry*, Peniscola 1985, Lecture Notes in Math. 1209 (Springer, Berlin, 1986) 168–183.
- [9] V.V. Goldberg, *Theory of Multicodimensional $(n + 1)$ -Webs* (Kluwer, Dordrecht, 1988).
- [10] C. LeBrun, *Anti-Self-Dual Riemannian 4-Manifolds*, Twistor theory (Plymouth), 81–94, Lecture Notes in Pure and Appl. Math., 169, Dekker, New York, 1995.
- [11] C.H. Taubes, The existence of anti-self-dual conformal structures, *J. Diff. Geom.* **36** (1992) 163–253.